MATH 240 **TEST I Solutions** 9/23/2002

Prof. John Beachy Show all of the work necessary to justify your answers.

1. (15 pts) (a) (12 pts) Use Gauss-Jordan reduction to solve the following linear system.

$$\begin{array}{rrrcr} x_1 & +2x_2 & +3x_3 & = & 6 \\ 2x_1 & -3x_2 & +2x_3 & = & 14 \\ 3x_1 & +x_2 & -x_3 & = & -2 \end{array}$$

See Example 12 on page 54 for a solution that avoids fractions.

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & \frac{4}{7} & -\frac{2}{7} \\ 0 & 1 & 2 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & \frac{13}{7} & \frac{46}{7} \\ 0 & 1 & \frac{4}{7} & -\frac{2}{7} \\ 0 & 1 & \frac{10}{7} & \frac{30}{7} \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} 1 & 0 & \frac{13}{7} & \frac{46}{7} \\ 0 & 1 & \frac{4}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(b) (3 pts) Explain why the answer to part (a) shows how to write the vector $\mathbf{b} = \begin{bmatrix} 6 \\ 14 \\ -2 \end{bmatrix}$ as a linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

The equation $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ -2 \end{bmatrix}$ gives the system in part (a), and so $\mathbf{b} = \mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3$.

2. (10 pts) Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

3. (15 pts) Write the matrix $A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$ as a product of elementary matrices. Then write A^{-1} as a product of elementary matrices. (*Hint:* You can do this without finding A^{-1}.)

Row reduce: $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The row reduction uses the matrices $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, so $E_3E_2E_1A = I$, and therefore $A = E_1^{-1}E_2^{-1}E_3^{-1}$ and $A^{-1} = E_3E_2E_1$.

$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

4. (15 pts) Show that if A is a nonsingular matrix and $AB = AC$ for matrices B, C, then $B = C$. Give an example to show that this can fail for 2×2 matrices if A is singular.

If $AB = AC$, then $A^{-1}(AB) = A^{-1}(AC)$, so $(A^{-1}A)B = (A^{-1}A)C$, or $IB = IC$, and therefore $B = C$.

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $AB = 0 = AC$ for $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. You could also take A to be the zero matrix.

5. (30 pts) Determine whether the given subset W is a subspace of the vector space V. (In each part, either check that all three of the necessary conditions hold, or give a numerical counterexample to one of them.)

(a) Let $V = \mathbf{R}^3$ and let $W = \{(x, y, z) \mid z = x + 2y\}$.

$0 = 0 + 2 \cdot 0$, so $(0, 0, 0) \in W$.

$$\begin{aligned}(a_1, a_2, a_1 + 2a_2) + (b_1, b_2, b_1 + 2b_2) &= (a_1 + b_1, a_2 + b_2, (a_1 + 2a_2) + (b_1 + 2b_2)) \\ &= (a_1 + b_1, a_2 + b_2, (a_1 + b_1) + 2(a_2 + b_2))\end{aligned}$$

$c(a_1, a_2, a_1 + 2a_2) = (ca_1, ca_2, c(a_1 + 2a_2)) = (ca_1, ca_2, ca_1 + 2a_2))$

(b) Let $V = \mathbf{R}^2$ and let $W = \{(x, y) \mid y \geq 0\}$. (the first and second quadrants)

The set is not closed under scalar multiplication, since $(0, 1) \in W$, but $(-1)(0, 1) = (0, -1) \notin W$.

(c) Let V be the vector space M_{44} of all 4×4 matrices, and let W be the set of all symmetric 4×4 matrices.

Recall that a matrix X is symmetric if and only if $X^T = X$.
The zero matrix belongs since $0^T = 0$.
If $A, B \in W$, then $A^T = A$ and $B^T = B$, so $(A + B)^T = A^T + B^T = A + B$, and therefore $A + B \in W$.
If $A \in W$, and c is any scalar, then $(cA)^T = cA^T = cA$, and so $cA \in W$.

6. (15 pts) In $\mathbf{R}^2$, use ordinary scalar multiplication $r \cdot (x, y) = (rx, ry)$ but define a new addition of vectors by $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, 2y_1 + y_2)$. Check each of these laws: (1) commutative law for addition, (2) associative law for addition, and (5), (6) distributive laws. If the law is valid, give a proof. If not, give a numerical counterexample to show that it fails.

(1) $(0, 1) \oplus (0, 0) = (0, 2)$
but $(0, 0) \oplus (0, 1) = (0, 1)$,
so the commutative law fails

(2) $((0, 1) \oplus (0, 2)) \oplus (0, 0) = (0, 4) \oplus (0, 0) = (0, 8)$
but $(0, 1) \oplus ((0, 2) \oplus (0, 0)) = (0, 1) \oplus (0, 4) = (0, 6)$,
so the associative law fails

(5) $c \cdot ((x_1, y_1) \oplus (x_2, y_2)) = c \cdot (x_1 + x_2, 2y_1 + y_2) = (cx_1 + cx_2, 2cy_1 + cy_2)$
and $c \cdot (x_1, y_1) \oplus c \cdot (x_2, y_2) = (cx_1, cy_1) \oplus (cx_2, cy_2) = (cx_1 + cx_2, 2cy_1 + cy_2)$,
so this law holds

(6) $(1 + 2) \cdot (0, 1) = 3 \cdot (0, 1) = (0, 3)$
but $1 \cdot (0, 1) \oplus 2 \cdot (0, 1) = (0, 1) \oplus (0, 2) = (0, 4)$,
so this law fails

3. Another algorithm for writing A as a product of elementary matrices:

$$\left[\begin{array}{rr|rr} 1 & -2 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|rr} 1 & -2 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{array}\right] \qquad [A|I] \rightsquigarrow [A_1|E_1] \quad E_1A = A_1 \quad A = E_1^{-1}A_1$$

$$\left[\begin{array}{rr|rr} 1 & -2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|rr} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{3} \end{array}\right] \qquad [A_1|I] \rightsquigarrow [A_2|E_2] \quad E_2A_1 = A_2 \quad A_1 = E_2^{-1}A_2$$

Answer:

$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \qquad A = E_1^{-1}A_1 = E_1^{-1}E_2^{-1}A_2$$

Grading scale: 85–97 A (6); 75–84 B (3); 60–74 C (8); 50–59 D (8); 34–49 F (3)