

Prof. John Beachy

Show all necessary work. No calculators.

1. (15 pts) For $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, find the rank and nullity of A , and find a basis for the nullspace of A .

Since there are 2 nonzero rows, the rank is 2, and therefore the nullity is $4 - 2 = 2$. The corresponding system of equations is $\begin{matrix} x_1 & +2x_2 & & -x_4 & = & 0 \\ & & x_3 & +3x_4 & = & 0 \end{matrix}$ so $x_1 = -2x_2 + x_4$ and $x_3 = -3x_4$. To find a basis, let $x_2 = r$ and

$x_4 = s$. The solution space of the system consists of vectors of the form $\begin{bmatrix} -2r + s \\ r \\ -3s \\ s \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$, and so the two given vectors form a basis for the solution space of the system, which is the nullspace of the matrix A .

2. (20 pts) Find a basis for the subspace of \mathbf{R}^4 spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -3 \\ 2 \\ -3 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

Put the vectors into a matrix, as the *columns* of the matrix, and row reduce.

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 3 & -3 & 1 \\ 1 & 0 & 2 & 2 \\ -1 & 1 & -3 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 3 & -3 & 1 \\ 0 & -2 & 2 & 3 \\ 0 & 3 & -3 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -1 & 4 \\ 0 & -2 & 2 & 3 \\ 0 & 3 & -3 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 & -9 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 11 \\ 0 & 0 & 0 & -10 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ The basis corresponds to the original vectors in columns 1, 2, and 4, since these columns contain}$$

the leading 1's. The set $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$ is a basis.

As an alternate solution, you can put the vectors in as the rows of a matrix. When you row reduce you do not change the row space, so you end up with the nonzero rows forming a basis. Doing it that way, at the end you have

to change rows to columns, and so you get a basis consisting of the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -14/11 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 13/11 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3/11 \end{bmatrix}$.

3. (20 pts) Let $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ and $T = \{(1, -1, 0), (0, 1, -1), (0, 0, 1)\}$ be ordered bases for \mathbf{R}_3 . Let $\mathbf{v} = (3, 2, 1)$. (a) Find the coordinate vector $[\mathbf{v}]_T$ of \mathbf{v} with respect to the basis T .

You can use trial and error to solve $(3, 2, 1) = _ (1, -1, 0) + _ (0, 1, -1) + _ (0, 0, 1)$ to get $[\mathbf{v}]_T = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$.

You can, of course, set up a system of equations to find the coordinates. As a third method, you could find the transition matrix $P_{T \leftarrow Std}$ from the standard basis to T . Then $[\mathbf{v}]_T = P_{T \leftarrow Std} \cdot [\mathbf{v}]_{Std} = P_{T \leftarrow Std} \cdot \mathbf{v}^T$. Note that $P_{T \leftarrow Std}$ is the inverse of the matrix whose columns are the vectors in T .

- (b) Find the transition matrix $P_{S \leftarrow T}$. Remember that you need to row reduce $[S|T] \rightsquigarrow [I|P_{S \leftarrow T}]$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

- (c) Use $P_{S \leftarrow T}$ to find the coordinate vector $[\mathbf{v}]_S$ of \mathbf{v} with respect to the basis S .

$$[\mathbf{v}]_S = P_{S \leftarrow T} \cdot [\mathbf{v}]_T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

4. (15 pts) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be linearly independent vectors in a vector space V . Let $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, let $\mathbf{w}_2 = \mathbf{v}_1 - \mathbf{v}_2$, and let $\mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_3$. Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent vectors in V .

You need to solve the equation $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$. Substituting gives $c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 - \mathbf{v}_2) + c_3(\mathbf{v}_1 + \mathbf{v}_3) = \mathbf{0}$. Rearranging you get $(c_1 + c_2 + c_3)\mathbf{v}_1 + (c_1 - c_2)\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. Now since the vectors \mathbf{v}_i are linearly independent, the coefficients must all be zero. You get the equations $c_3 = 0$, $c_1 - c_2 = 0$, and $c_1 + c_2 + c_3 = 0$, and it is then easy to see that $c_3 = 0$, $c_2 = 0$, and $c_1 = 0$, which proves that the \mathbf{w}_i 's are linearly independent.

Alternate solution: Since the \mathbf{v}_i 's form a basis for a subspace, you can find the coordinate vectors of the \mathbf{w}_i 's relative to this basis. These coordinate vectors are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, respectively. Putting the coordinate vectors into a matrix and row reducing will show that they are linearly independent, and this answers the question.

5. (10 pts) Use a 3×3 determinant to find the values of c for which the vectors $[-1 \ 0 \ -1]$, $[2 \ 1 \ 2]$, and $[1 \ 1 \ c]$ are linearly independent in \mathbf{R}_3 . Solution: Put the vectors into a matrix and find the determinant.

$\begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & c \end{vmatrix} = \begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & c-1 \end{vmatrix} = (c-1)$ by using one elementary row operation. Then the vectors are linearly independent if and only if the determinant is nonzero. The final answer is that the vectors are linearly independent if and only if $c \neq 1$.

6. (10 pts) Let A be an invertible $n \times n$ matrix. Explain why $\det(\text{adj}(A)) = [\det(A)]^{n-1}$.

The basic relationship is that $A \cdot \text{adj}(A) = \det(A) \cdot I_n$. Take the determinant of both sides to get

$$\begin{aligned} \det(A \cdot \text{adj}(A)) &= \det(\det(A) \cdot I_n) \\ \det(A) \cdot \det(\text{adj}(A)) &= \det(A)^n \cdot \det(I_n) \\ \det(A) \cdot \det(\text{adj}(A)) &= \det(A)^n \end{aligned}$$

Now since A is invertible, you have $\det(A) \neq 0$, and so you can divide both sides of the last equation by $\det(A)$. This gives the equation that you are looking for.

Remember that $\det(cI_n) = c^n$, because you have a c in each of n rows, and therefore a c is factored out n times.

7. (10 pts) Find the adjoint of $A = \begin{bmatrix} 1 & a & c \\ b & ab & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Check that $A \cdot \text{adj}(A) = \det(A) \cdot I_3$.

Since the determinant has the block form $\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$, you can evaluate it as $|X| \cdot |Z| = (ab - ba)(1) = 0$.

Calculating the matrix of cofactors gives $\text{adj}(A) = \begin{bmatrix} ab & -b & 0 \\ -a & 1 & 0 \\ -abc & bc & 0 \end{bmatrix}^T = \begin{bmatrix} ab & -a & -abc \\ -b & 1 & bc \\ 0 & 0 & 0 \end{bmatrix}$ and then it is easy

to check that $A \cdot \text{adj}(A)$ is the zero matrix.

Grades: 85-98 A (5); 75-84 B (8); 61-72 C (12); 54 D (1)

The class average was 75.1. At the end of the semester I look for natural breaks in the scores, so missing a letter grade by 1 or 2 points on one exam probably won't make difference in the final grade.