MATH 240 **Solutions to the sample questions** for Exam 3 on 11/30/2007

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1. Let $L : \mathbf{R}^5 \to \mathbf{R}^4$ be the linear transformation defined by $L(\mathbf{x}) = A\mathbf{x}$, for the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 & -1 \\ 1 & 0 & 0 & 2 & -1 \\ 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}. \text{ Given that } A \text{ row-reduces to } \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

(a) find a basis for $\ker L$;

Since $\ker L$ is the solution space of the equation $L(\mathbf{x}) = \mathbf{0}$, we need to find the solution space of the system of equations $A\mathbf{x} = \mathbf{0}$. After row-reducing the matrix, we have the corresponding equations $x_1 = -2x_4$; $x_3 = x_4$; $x_5 = 0$.

Choose $x_2 = 1$, $x_4 = 0$ and then $x_2 = 0$, $x_4 = 1$ to obtain the basis $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

(b) find a basis for $\operatorname{range} L$;

Since $\operatorname{range} L$ is the column space of A, as basis we take the columns of A that correspond to the leading 1's in the reduced matrix, giving us the basis $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$.

(c) find $\dim(\ker L)$ and $\dim(\operatorname{range} L)$.

We have $\dim(\ker L) = 2$ and $\dim(\operatorname{range} L) = 3$. Check: $2 + 3 = \dim(\mathbf{R}^5)$.

2. Define the linear transformation $L : P_2 \to P_2$ by $L(p(t)) = p(t) + 2p'(t)$.

(a) Find the matrix $M_{S \leftarrow S}(L)$ of L relative to the standard basis $S = \{t^2, t, 1\}$.

For $p(t) = at^2 + bt + c$, we have $L(p(t)) = at^2 + bt + c + 2(2at + b) = at^2 + (4a + b)t + (2b + c)$. Using coordinate matrices, we have

$$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a \\ 4a + b \\ 2b + c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad M_{S \leftarrow S}(L) = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(b) Find the matrix $M_{T \leftarrow T}(L)$ of L relative to the basis $T = \{t^2 + t + 1, t + 1, 1\}$.
We need to find the coordinate vectors of the images of the basis vectors.

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The coordinate vectors we have found go into the matrix as columns. $M_{T \leftarrow T}(L) = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$

You could also use transition matrices: $M_{T \leftarrow T}(L) = P_{T \leftarrow S} \cdot M_{S \leftarrow S}(L) \cdot P_{S \leftarrow T} =$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

3. Let $L : \mathbf{R}_3 \to \mathbf{R}_3$ be the linear transformation defined by $L(x_1, x_2, x_3) = (x_1,\, 2x_2 + x_3,\, x_2 + 2x_3)$. Let $S = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$ be the standard basis for $\mathbf{R}_3$, and let T be the basis $\{(1,0,0),\ (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),\ (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$.

(a) Find the matrix representation $M_{S\leftarrow S}(L)$ of L with respect to the basis S.

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2x_2 + x_3 \\ x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad M_{S\leftarrow S}(L) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

(b) Find the matrix representation $M_{T\leftarrow T}(L)$ of L with respect to the basis T.

$$[\,L(T)\,] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$[\,T \mid L(T)\,] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 0 & 3 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad M_{T\leftarrow T}(L) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notes: If you realize that $L(\mathbf{v}_1) = \mathbf{v}_1, L(\mathbf{v}_2) = 3\mathbf{v}_2, L(\mathbf{v}_3) = \mathbf{v}_3$, for the new basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, then you can immediately write down the matrix $M_{T\leftarrow T}(L)$. Actually, the new basis was chosen to do exactly this.

The matrix came from Exercise 15 in Section 8.8. You now have enough background to read Sections 8.7 and 8.8, which give some interesting applications of the theory of similar matrices in geometry. For certain curves in $\mathbf{R}^2$ and surfaces in $\mathbf{R}^3$, finding a similar diagonal matrix allows us to put a curve or surface into standard form by choosing an appropriate basis. Diagonalizing this particular matrix allows us to see that the quadric surface $x^2 + 2y^2 + 2z^2 + 2yz = 1$ is an ellipsoid which is written in standard form as

$$\frac{x'^2}{1} + \frac{y'^2}{1} + \frac{z'^2}{1/3} = 1$$

after making the substitution $x = -y'$, $y = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}z'$, $z = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}z'$.

4. (10 pts; 4.5 #7) Show that if the $n \times n$ matrix B is similar to matrix A, then B^T is similar to A^T.

If B is similar to A, then $B = P^{-1}AP$ for some invertible matrix P. Taking the transpose gives $B^T = (P^{-1}AP)^T = P^TA^T(P^{-1})^T$. Since $P^T(P^{-1})^T = (P^{-1}P)^T = I^T = I$, we can see that P^T is the inverse of $(P^{-1})^T$. Thus $B^T = X^{-1}A^TX$ for $X = (P^{-1})^T$, and this shows that B^T is similar to A^T.

5. (15 pts) Let W be the subspace of R_4 spanned by the vectors $(1,-1,1,1)$ and $(1,0,2,1)$. Use the Gram-Schmidt process to find an orthonormal basis for W.

Let $\mathbf{u}_1 = (1,-1,1,1)$ and $\mathbf{u}_2 = (1,0,2,1)$.

Then $\mathbf{v_1} = (1,-1,1,1)$, and $\mathbf{v}_2$ is $\mathbf{u}_2$ minus its projection onto $\mathbf{v}_1$.

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1,0,2,1) - \frac{(1+0+2+1)}{(1+1+1+1)}(1,-1,1,1) = (1,0,2,1) - (1,-1,1,1) = (0,1,1,0).$$

Finally, divide each vector by its length: $\mathbf{w}_1 = \dfrac{1}{2}(1,-1,1,1)$ and $\mathbf{w}_2 = \dfrac{1}{\sqrt{2}}(0,1,1,0)$.

6. Let M_{22} be the vector space of all 2×2 matrices. For A, B in M_{22}, define an inner product by $(A,B) = \operatorname{tr}(B^TA)$. (*Recall:* $\operatorname{tr}(A)$ denotes the trace of A, which is the sum of entries on the main diagonal.)

(a) Check that $(A,B) = (B,A)$ for all A, B in M_{22}.

$(A,B) = \operatorname{tr}(B^TA) = \operatorname{tr}((B^TA)^T) = \operatorname{tr}(A^TB) = (B,A)$ since $\operatorname{tr}(X^T) = \operatorname{tr}(X)$.

(b) For any 2×2 matrix A, check that $(A,A) = 0$ if and only if $A = 0$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $(A,A) = \operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}\right) = a^2+b^2+c^2+c^2$, and this is zero if and only if each of a, b, c, d is zero.

Note: you could use this computational approach to check part (a).

7. (10 pts) Answer EITHER part **A** OR part **B**.

A. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space V, prove the *parallelogram law*, which states that $||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$.

Proof:

$||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = (\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v})$
$= (\mathbf{u}, \mathbf{u}) + 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) + (\mathbf{u}, \mathbf{u}) - 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) = 2(\mathbf{u}, \mathbf{u}) + 2(\mathbf{v}, \mathbf{v}) = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$.

B. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthogonal set of nonzero vectors in an inner product space V. Show that S is a linearly independent set.

Solution: See the proof of Theorem 5.4 on page 316. Remember that to show that a set of vectors is linearly independent you set up the equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$ and prove that all of the c_i's are zero. You have to use the inner product and the fact that the vectors are orthogonal to each other to get each $c_i = 0$.

8. Find the orthogonal complement in $\mathbf{R}^4$ of the subspace W spanned by the vectors $(1, 1, 0, 0)$, $(0, 1, 1, 0)$, $(0, 0, 1, 1)$.

We need to find all vectors (x_1, x_2, x_3, x_4) that are orthogonal to the 3 given basis vectors. Since we can check this with the dot product, this is equivalent to solving the system whose matrix consists of the given rows.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim> \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim> \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad x_1 = -x_4;\ x_2 = x_3;\ x_3 = -x_4$$

Therefore $(-1, 1, -1, 1)$ is a basis for the orthogonal complement. Check: $\dim(W) + \dim(W^{\perp}) = 3 + 1 = 4$