MATH 240 | **EXAM III** | 11/30/2007

Prof. John Beachy | Show all necessary work | No calculators

1. (10 pts; §6.2 Example 10) Let $L : P_2 \to P_2$ be the linear transformation defined by $L(at^2 + bt + c) = (a + 2b)t + (b + c)$. (a) Find a basis for $\ker(L)$. (b) Find a basis for range(L).

2. (10 pts; §6.2 #7) Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$. Let $L : M_{23} \to M_{33}$ be the linear transformation defined by $L(X) = AX$ for all matrices X in M_{23}. (a) Find the dimension of $\ker(L)$. (b) Find the dimension of range(L).

3. (15 pts; §5.4 #15) Find an orthonormal basis for the subspace of $\mathbf{R}^3$ consisting of all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $a + b + c = 0$. *Hint:* Choose any basis and then use the Gram-Schmidt process.

4. (10 pts; §5.5 #15,19) Let $\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}$, and $\mathbf{w}_3 = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$. Let W be the subspace spanned by the set $\{\mathbf{w}_1, \mathbf{w}_2\}$. and let $\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

   (a) Write $\mathbf{v} = \mathbf{w} + \mathbf{u}$, with $\mathbf{w}$ in W and $\mathbf{u}$ in $W^\perp$. (b) Find the distance from $\mathbf{v}$ to W.

5. (25 pts; §6.3 #1) Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 2u_2 \\ 2u_1 - u_2 \end{bmatrix}$. Let S be the standard basis for $\mathbf{R}^2$, and let $T = \{\mathbf{w}_1, \mathbf{w}_2\}$, for $\mathbf{w}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

   (a) Find the matrix representation $M_{S \leftarrow S}(L)$ of L with respect to S.

   $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 2u_2 \\ 2u_1 - u_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. *(Fill in the entries of the matrix.)*

   (b) Find the matrix representation $M_{T \leftarrow T}(L)$ of L with respect to T by calculating $[L(\mathbf{w}_1)]_T$ and $[L(\mathbf{w}_2)]_T$.

   (c) (§6.5 #3) Find the matrix representation $M_{T \leftarrow T}(L)$ of L with respect to T using transition matrices.

6. (30 pts) **Work 3 of the following 4 problems.**

   A. (§6.5 #17) Prove that if A and B are similar matrices, then $\det(A) = \det(B)$.

   B. (§6.2 #20) Let L be a linear transformation, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors in V. Prove that if $T = \{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)\}$ is linearly independent, then so is S.

   C. (§5.3 #19) Let V be an inner product space. Prove that if $\mathbf{u}$ and $\mathbf{v}$ are any vecors in V, then $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ if and only if $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

   D. (§5.3 #23) Let V be an inner product space, and let $\mathbf{u}$ be a fixed vector in V. Prove that the set of all vectors in V that are orthogonal to $\mathbf{u}$ is a subspace of V.