MATH 240 **FINAL EXAM** **12/13/2000**

1. (30 pts) Let $A = \begin{bmatrix} 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 3 \\ 2 & 2 & -1 & 4 & -4 \\ -3 & -3 & -1 & -1 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Show that B is the reduced row echelon form for A.
(b) Find a basis for the column space of A.
(c) Find a basis for the row space of A.
(d) Find a basis for the nullspace of A.
(e) Let $L : \mathbf{R}^5 \to \mathbf{R}^4$ be the linear transformation defined by $L(\mathbf{v}) = A\mathbf{v}$, for all $\mathbf{v}$ in $\mathbf{R}^5$.
Is L a one-to-one transformation? Explain your answer.

2. (20 pts) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 0 & -4 \\ -1 & 1 & 5 \end{bmatrix}$.

(a) Find the characteristic polynomial of A. *Hint*: One of the factors is $\lambda - 2$.
(b) Can the matrix A be diagonalized? Explain your answer.

3. (20 pts) Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

(a) Show that the characteristic polynomial of A is $(\lambda - 3)^2(\lambda + 2)$.
(b) Can the matrix A be diagonalized? Explain your answer.

4. (20 pts) Let P_2 be the vector space of all polynomials of degree at most 2. Define the function $L : \mathrm{P}_2 \to \mathrm{P}_2$ by $L(p(x)) = p(x+1) - p(x)$, for all polynomials $p(x)$ in P_2.

(a) Find the matrix of L relative to the standard basis $S = \{x^2, x, 1\}$.
(b) Find the rank of L.
(c) Find the kernel (or nullspace) of L.

5. (20 pts) Let W be the subspace of $\mathbf{R}_4$ spanned by $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$.

(a) Find an orthonormal basis for W.
(b) Use your answer in (a) to find the projection of the vector $(0, 2, -1, 0)$ on W.

6. (30 pts) Define a linear transformation $L : \mathbf{R}_2 \to \mathbf{R}_2$ by $L(x, y) = (x + \sqrt{3}y,\ \sqrt{3}x - y)$, for all (x, y) in $\mathbf{R}_2$. Let S be the standard basis $\{(1, 0),\ (0, 1)\}$, and let T be the basis $\{\,(\frac{1}{2},\ -\frac{\sqrt{3}}{2}),\ (\frac{\sqrt{3}}{2},\ \frac{1}{2})\}$.

(a) Find the matrix representation A of L relative to the standard basis S.
(b) Find the matrix representation B of L relative to the basis T.
(c) Show that the matrix $P = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ is orthogonal.
(d) Show that $B = P^{-1}AP$. (You may explain why the equation is true **OR** do the computation.)

7. (30 pts) For each of the following subsets, decide whether or not the subset is a subspace of the given vector space. If it *is* a subspace, show that it satisfies the necessary conditions. If it is *not* a subspace, explain why not.

(a) $\{(x, y) \mid x^2 + y^2 \le 1\}$ in $\mathbf{R}_2$.
(b) $\{p(x) \mid p(x+1) = p(x)\}$ in the vector space P_2 of all polynomials of degree at most 2.
(c) The set of all symmetric $n \times n$ matrices in the vector space M_{nN} of all $n \times n$ matrices.

8. (30 pts) For each of the following questions, write out a careful proof, in complete sentences. Even if you cannot complete the proof, you will get partial credit for stating the relevant definitions.

(a) Prove that if A is an invertible 3×3 matrix, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a set of column vectors that forms a basis for $\mathbf{R}^3$, then the set $\{A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3\}$ is a basis for $\mathbf{R}^3$.

(b) Prove that if $\mathbf{x}$ and $\mathbf{y}$ are any vectors in an inner product space V, then the vector $\mathbf{u} = ||\mathbf{y}||\,\mathbf{x} + ||\mathbf{x}||\,\mathbf{y}$ is orthogonal to the vector $\mathbf{v} = ||\mathbf{y}||\,\mathbf{x} - ||\mathbf{x}||\,\mathbf{y}$.

(c) Let A and B be $n \times n$ matrices. Prove that the nullspace of B is contained in the nullspace of AB. Prove that the column space of AB is contained in the column space of A.