Math 240, Draft Final Exam, May 11 , 2006

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(1) (20 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & 0 & 4 \\ 1 & 2 & 1 & 1 & 1 \\ 4 & 5 & 5 & 2 & 8 \end{bmatrix}.$$

(a) Find a basis for the column space of A.

(b) Find a basis for the nullspace of A.

(c) Let $L : R^5 \to R^3$ be the linear transformation that has A as its standard matrix. What is the dimension of the range of L?

(2) (15 points) Let P_2 be the vector space of all polynomials in the variable t of degree 2 or less. In P_2, is the following set linearly independent?

$$\{2t^2 + 3t + 4, t^2 + t + 1, 3t^2 + 4t + 5\}$$

(3) (15 points) Let P_2 be the vector space of all polynomials in the variable t of degree 2 or less. Let S be the ordered basis $\{t^2, t, 1\}$ for P_2. Let $L : P_2 \to P_2$ be the linear transformation such that for all $p(t)$ in P_2, $L(p(t)) = p(t) + p'(t)$. Find A, the matrix representation for L with respect to S.

(4) (15 points) Find the eigenvalues of the following matrix.

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

(5) (15 points) Consider the matrix

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Given: The eigenvalues of A are 3 and 5.
If possible, find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

(6) (20 points) Let A be an 5×5 matrix of rank 4. Let $L : R^5 \to R^5$ be the linear transformation such that $L(v) = Av$ for all v in R^5.
Answer the following questions. Include a brief explanation with each answer.

(a) What is the dimension of the kernel of L?

(b) Is $\det A = 0$?

(c) Are there vectors v and w in R^5 such that $v \neq w$ but $L(v) = L(w)$?

(d) Is 0 an eigenvalue of A ?

(7) (15 points) Is it true that for all $n \times n$ matrices A and B,

$$A^2 + 2AB + B^2 = (A + B)^2 \quad ?$$

Explain.

(8) (10 points) Let V be the vector space of all 2×2 matrices. For each of the following subsets of W of V, determine whether W is a subspace of V.

(a) W is the set of all A in V such that A is invertible.

(b) W is the set of all A in V such that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A - A \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(9) (15 points) Let V be the vector space of all 2×2 matrices A such that $A = -A^T$.
(a) Find a basis for V.

(b) What is the dimension of V?

(10) (15 points) Let V be a vector space with an inner product. Let v and w be vectors in V. Assume that v and w are orthogonal. Prove that

$$\| v + w \| = \| v - w \|$$

(11) (15 points) Let A be an $n \times n$ matrix. Assume that $A^2 = 2A$. Prove that if λ is an eigenvalue for A, then $\lambda = 0$ or $\lambda = 2$.

(12) (15 points) Let A be an $m \times n$ matrix. Assume that the dimension of the nullspace of A is 0. Let $\{v_1, v_2, \ldots, v_n\}$ be a linearly independent set of vectors in R^m.
Prove that the set $\{Av_1, Av_2, \ldots, Av_n\}$ is linearly independent.