## Math 240 FINAL EXAM (12/9/09)

(1) Find the general solution to the system of equations

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y + 4z &= 0 \\ 5x + 6y + 7z &= 3 \end{aligned}$$

(2) Let $V = \mathbb{R}_3$, and define a product on $V$ by $([x_1 \;\; y_1 \;\; z_1], [x_2 \;\; y_2 \;\; z_2]) = x_1x_2 + y_1y_2$. Show that this is not an inner product by finding a property of inner products that fails to hold.

(3) Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 0 & 7 \\ 1 & 0 & 1 & 1 & -1 \end{bmatrix}$$

(a) Find a basis for the null-space of $A$.
(b) Find a basis for the column-space of $A$.
(c) Find a basis for the row-space of $A$.
(d) What is the rank of $A$? Explain.

(4) Let $L : V \to W$ be a linear transformation.
(a) Prove that $L(\mathbf{0}) = \mathbf{0}$.
(b) Prove that the range of $L$ is a subspace of $W$.

(5) Let $V$ be an inner product space, and let $\mathbf{u}, \mathbf{v}$ be vectors in $V$. Prove that $\| \mathbf{u} + \mathbf{v} \|^2 + \| \mathbf{u} - \mathbf{v} \|^2 = 2\| \mathbf{u} \|^2 + 2\| \mathbf{v} \|^2$.

(6) Let $V$ be an inner product space, let $\mathbf{w}_1, \mathbf{w}_2$ be linearly independent vectors in $V$, and let $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$. Let $\mathbf{u}$ be a nonzero vector in $V$ which is an element of $W^{\perp}$, the orthogonal complement of $W$ in $V$. Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}$ are linearly independent.

(7) Let $V = P_3$, the space of all polynomials of degree $\leq 3$. Define an inner product on $V$ by $(p(t), q(t)) = \int_0^1 p(t)q(t)dt$. Let $W$ be the subspace of $V$ with basis $\{t, t^2\}$. Find an orthogonal basis for $W$ (you do not need to find an orthonormal basis).

(8) Let $V = M_{22}$, and let $W = \mathbb{R}^2$. Define a function $L : V \to W$ as follows. Let

$$\mathbf{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and for any matrix $A$ in $V$, define $L(A) = A\,\mathbf{b}$.
(a) Prove that $L$ is a linear transformation.
(b) Find the representation of $L$ with respect to the standard basis for $V$:

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and the standard basis for $W$:

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

(9) Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -4 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

(a) Find the eigenvalues of $A$.
(b) For each eigenvalue of $A$, find a corresponding eigenvector.
(c) Find the eigenvalues of $A^2$.

(10) Let

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 3 & 10 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 2 & -5 & 2 \end{bmatrix}$$

(a) Show that the determinant of $A$ is equal to 100.
(b) Use the result of part (a) to show that $A^3 \neq 2A^2$.

(11) Let $V = P_2$, the vector space of all polynomials of degree at most 2. Let $S = \{1+t, 1-t, t^2\}$ and $T = \{1, 1+t, 1+t^2\}$ be ordered bases for $V$. Let $p(t)$ be a polynomial such that

$$[p(t)]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) Find $p(t)$
(b) Find the transition matrix $P_{T \leftarrow S}$ between $S$ and $T$.
(c) Find $[p(t)]_T$.

(12) Let $L : V \to W$ be a linear transformation with $\ker L = \{\mathbf{0}\}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be linearly independent vectors in $V$. Prove that the vectors $L(\mathbf{v}_1), \ldots L(\mathbf{v}_n)$ are also linearly independent.