**Math 240 FINAL EXAM (12/9/09) - SOLUTIONS**

(1) Find the general solution to the system of equations

$$\begin{aligned} x+y+z&=1\\ 2x+3y+4z&=0\\ 5x+6y+7z&=3 \end{aligned}$$

**SOLUTION:**

$$\begin{bmatrix} 1 & 1 & 1 & 1\\ 2 & 3 & 4 & 0\\ 5 & 6 & 7 & 3 \end{bmatrix} \rightarrow_{\binom{r_2-2r_1\rightarrow r_2}{r_3-5r_1\rightarrow r_3}} \begin{bmatrix} 1 & 1 & 1 & 1\\ 0 & 1 & 2 & -2\\ 0 & 1 & 2 & -2 \end{bmatrix} \rightarrow_{\binom{r_3-r_2\rightarrow r_3}{r_1-r_2\rightarrow r_1}} \begin{bmatrix} 1 & 0 & -1 & 3\\ 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x=z+3$, $y=-2z-2$ and $z$ is free.

(2) Let $V=\mathbb{R}_3$, and define a product on $V$ by $([x_1 \quad y_1 \quad z_1],[x_2 \quad y_2 \quad z_2]) = x_1x_2+y_1y_2$. Show that this is not an inner product by finding a property of inner products that fails to hold.
**SOLUTION:**
The property that fails to hold is $(\mathbf{u},\mathbf{u})=0 \iff \mathbf{u}=0$.
For example, if $\mathbf{u}=[0 \quad 0 \quad 1]$ then $\mathbf{u}\neq\mathbf{0}$ but $(\mathbf{u},\mathbf{u})=0$.

(3) Let

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 & 3\\ 1 & 2 & 1 & 0 & 7\\ 1 & 0 & 1 & 1 & -1 \end{bmatrix}$$

(a) Find a basis for the null-space of $A$.
**SOLUTION:**
Reduce the augmented matrix $[A|0]$ to RREF:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 0\\ 1 & 2 & 1 & 0 & 7 & 0\\ 1 & 0 & 1 & 1 & -1 & 0 \end{bmatrix} \rightarrow_{\binom{r_2-r_1\rightarrow r_2}{r_3-r_1\rightarrow r_3}} \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 0\\ 0 & 1 & 0 & -1 & 4 & 0\\ 0 & -1 & 0 & 0 & -4 & 0 \end{bmatrix} \rightarrow_{\binom{r_1-r_2\rightarrow r_1}{r_3+r_2\rightarrow r_3}} \begin{bmatrix} 1 & 0 & 1 & 2 & -1 & 0\\ 0 & 1 & 0 & -1 & 4 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\rightarrow_{-r_3\rightarrow r_3} \begin{bmatrix} 1 & 0 & 1 & 2 & -1 & 0\\ 0 & 1 & 0 & -1 & 4 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow_{\binom{r_1-2r_3\rightarrow r_1}{r_2+r_3\rightarrow r_2}} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

So the solutions are

$$\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3+x_5\\ -4x_5\\ x_3\\ 0\\ x_5 \end{bmatrix} = x_3\begin{bmatrix} -1\\ 0\\ 1\\ 0\\ 0 \end{bmatrix} + x_5\begin{bmatrix} 1\\ -4\\ 0\\ 0\\ 1 \end{bmatrix}$$

so $\{[-1\;0\;1\;0\;0]^T,[1\;-4\;0\;0\;1]^T\}$ is a basis for the null-space of $A$.

(b) Find a basis for the column-space of $A$.
**SOLUTION:**
From above, the RREF of $A$ has leading ones in columns 1, 2 and 4. Therefore the first, second and fourth columns of $A$ form a basis: $\{[1\;1\;1]^T,[1\;2\;0]^T,\{[1\;0\;1]^T\}$.

(c) Find a basis for the row-space of $A$.
**SOLUTION:**
From above, the three rows of the RREF of $A$ have leading ones, so they form a basis for the row-space of $A$:
$\{[1\;0\;1\;2\;-1],[0\;1\;0\;-1\;4],[0\;0\;0\;1\;0]\}$.

(d) What is the rank of $A$? Explain.
**SOLUTION:**
rank $A=3$, since by part (b) that is the dimension of the column-space. (or, since by part (c) that is the dimension of the row-space).

(4) Let $L : V \to W$ be a linear transformation.

(a) Prove that $L(\mathbf{0}) = \mathbf{0}$.

**SOLUTION:**

$L(\mathbf{0}) = L(0 \cdot \mathbf{0}) = 0 \cdot L(\mathbf{0}) = \mathbf{0}$.

**OR:**

$L(\mathbf{0}) = L(\mathbf{0} + (-1) \cdot \mathbf{0}) = L(\mathbf{0}) + (-1) \cdot L(\mathbf{0}) = L(\mathbf{0}) - L(\mathbf{0}) = \mathbf{0}$.

(b) Prove that the range of $L$ is a subspace of $W$.

**SOLUTION:**

Let $\mathbf{w}_1, \mathbf{w}_2 \in \operatorname{range} L$. Then $\mathbf{w}_1 = L(\mathbf{v}_1)$ for some $\mathbf{v}_1 \in V$, and $\mathbf{w}_2 = L(\mathbf{v}_2)$ for some $\mathbf{v}_2 \in V$.

So $\mathbf{w}_1 + \mathbf{w}_2 = L(\mathbf{v}_1) + L(\mathbf{v}_2) = L(\mathbf{v}_1 + \mathbf{v}_2) \in \operatorname{range} L$ since $\mathbf{v}_1 + \mathbf{v}_2 \in V$.

Also, let $\mathbf{w} \in \operatorname{range} L$, $c \in \mathbb{R}$, then $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$, so

$c \cdot \mathbf{w} = c \cdot L(\mathbf{v}) = L(c \cdot \mathbf{v}) \in \operatorname{range} L$ since $c\,\mathbf{v} \in V$.

Thus the range is closed under addition and under scalar multiplication, so it is a subspace.

(5) Let $V$ be an inner product space, and let $\mathbf{u}, \mathbf{v}$ be vectors in $V$. Prove that

$\| \mathbf{u} + \mathbf{v} \|^2 + \| \mathbf{u} - \mathbf{v} \|^2 = 2\| \mathbf{u} \|^2 + 2\| \mathbf{v} \|^2$

**SOLUTION:**

$\| \mathbf{u} + \mathbf{v} \|^2 + \| \mathbf{u} - \mathbf{v} \|^2 = (\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v})$

$= (\mathbf{u}, \mathbf{u}) + (\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{u}) + (\mathbf{v}, \mathbf{v}) + (\mathbf{u}, \mathbf{u}) - (\mathbf{u}, \mathbf{v}) - (\mathbf{v}, \mathbf{u}) + (\mathbf{v}, \mathbf{v})$

$= 2(\mathbf{u}, \mathbf{u}) + 2(\mathbf{v}, \mathbf{v}) = 2\| \mathbf{u} \|^2 + \| \mathbf{v} \|^2$

(6) Let $V$ be an inner product space, let $\mathbf{w}_1, \mathbf{w}_2$ be linearly independent vectors in $V$, and let $W = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$. Let $\mathbf{u}$ be a nonzero vector in $V$ which is an element of $W^\perp$, the orthogonal complement of $W$ in $V$. Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}$ are linearly independent.

**SOLUTION:**

Supose $c_1\,\mathbf{w}_1 + c_2\,\mathbf{w}_2 + c_3\,\mathbf{u} = \mathbf{0}$.

Then $(\mathbf{u}, c_1\,\mathbf{w}_1 + c_2\,\mathbf{w}_2 + c_3\,\mathbf{u}) = (\mathbf{u}, \mathbf{0}) = 0$. So

$c_1(\mathbf{u}, \mathbf{w}_1) + c_2(\mathbf{u}, \mathbf{w}_2) + c_3(\mathbf{u}, \mathbf{u}) = 0$. But since $\mathbf{u} \in W^\perp$, $(\mathbf{u}, \mathbf{w}_1) = (\mathbf{u}, \mathbf{w}_2) = 0$, so

$c_3(\mathbf{u}, \mathbf{u}) = 0$, and since $\mathbf{u} \neq \mathbf{0}$ this implies $c_3 = 0$.

Therefore $c_1\,\mathbf{w}_1 + c_2\,\mathbf{w}_2 = \mathbf{0}$. Since $\mathbf{w}_1, \mathbf{w}_2$ are linearly independent, this implies $c_1 = c_2 = 0$.

Thus $c_1 = c_2 = c_3 = 0$, proving that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}$ are linearly independent.

(7) Let $V = P_3$, the space of all polynomials of degree $\leq 3$. Define an inner product on $V$ by $(p(t), q(t)) = \int_0^1 p(t)q(t)dt$.

Let $W$ be the subspace of $V$ with basis $\{t, t^2\}$. Find an orthogonal basis for $W$ (you do not need to find an orthonormal basis).

( *Hint:* $\int_0^1 t^n dt = \frac{1}{n+1}$ )

**SOLUTION:**

We use the GS process: $p_1(t) = t$, and $p_2(t) = t^2 - \frac{(t^2,t)}{(t,t)}t = t^2 - \frac{\int_0^1 t^3 dt}{\int_0^1 t^2 dt}t = t^2 - \frac{1/4}{1/3}t = t^2 - \frac{3}{4}t$.

So $\{t, t^2 - \frac{3}{4}t\}$ is an orthogonal basis for $W$.

(8) Let $V = M_{22}$, and let $W = \mathbb{R}^2$. Define a function $L : V \to W$ as follows.
Let

$$\mathbf{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and for any matrix $A$ in $V$, define $L(A) = A\,\mathbf{b}$.

(a) Prove that $L$ is a linear transformation.
**SOLUTION:**
Let $A_1, A_2 \in V$. Then $L(A_1 + A_2) = (A_1 + A_2)\,\mathbf{b} = A_1\,\mathbf{b} + A_2\,\mathbf{b} = L(A_1) + L(A_2)$.
Also, if $A \in V, c \in \mathbb{R}$ then $L(cA) = (cA)\,\mathbf{b} = c(A\,\mathbf{b}) = cL(A)$. So $L$ is linear.

(b) Find the representation of $L$ with respect to the standard basis for $V$:

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and the standard basis for $W$:

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

**SOLUTION:**

$$L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix},$$

$$L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix},$$

So the matrix is

$$\begin{bmatrix} 3 & 5 & 0 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$

(9) Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -4 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

(a) Find the eigenvalues of $A$.
**SOLUTION:**
The characteristic polynomial is

$$\det(tI - A) = \det \begin{bmatrix} t-1 & 0 & 1 \\ -2 & t+4 & -2 \\ -2 & 2 & t \end{bmatrix}$$

$$= (t-1)(t+4)t + 0 + 1(-2)(2) - 1(t+4)(-2) - 0 - (t-1)(-2)(2)$$
$$= t^3 + 3t^2 - 4t - 4 + 2t + 8 + 4t - 4 = t^3 + 3t^2 + 2t = t(t+2)(t+1).$$

So the eigenvalues are $0, -1, -2$.

(b) For each eigenvalue of $A$, find a corresponding eigenvector. For the eigenvalue $\lambda = 0$, we find the nullspace of $0I - A$:

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 4 & -2 \\ -2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & -4 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x = z$, $y = z$ and $z$ is free, so $[1\ 1\ 1]^T$ is a corresponding eigenvector. For the eigenvalue $\lambda = -1$, we find the nullspace of $-I - A$:

$$\begin{bmatrix} -2 & 0 & 1 \\ -2 & 3 & -2 \\ -2 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} -2 & 0 & -1 \\ 0 & 3 & -3 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x = \frac{1}{2}z$, $y = z$ and $z$ is free, so $[1\ 2\ 2]^T$ is a corresponding eigenvector.
For the eigenvalue $\lambda = -2$, we find the nullspace of $-2I - A$:

$$\begin{bmatrix} -3 & 0 & 1 \\ -2 & 2 & -2 \\ -2 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

So $x = \frac{1}{3}z$, $y = \frac{4}{3}z$ and $z$ is free, so $[1\ 4\ 3]^T$ is a corresponding eigenvector.

(c) Find the eigenvalues of $A^2$.
**SOLUTION:**
If $A\,\mathbf{v} = \lambda\,\mathbf{v}$ then $A^2\,\mathbf{v} = A(\lambda\,\mathbf{v}) = \lambda^2\,\mathbf{v}$. Therefore the eigenvalues of $A^2$ are $0^2 = 0$ $(-1)^2 = 1$ and $(-2)^2 = 4$.

(10) Let

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 3 & 10 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 2 & -5 & 2 \end{bmatrix}$$

(a) Show that the determinant of $A$ is equal to 100.
**SOLUTION:**
One option is to make one row-reduction step ($r_4 - r_1 \to r_4$) and then expand along the first column:

$$\det(A) = \det \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 3 & 10 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -10 & 2 \end{bmatrix} = \det \begin{bmatrix} 3 & 10 & 0 \\ 1 & 0 & 4 \\ 0 & -10 & 2 \end{bmatrix} =$$

$0 + 0 + 0 - 0 - 10(1)(2) - (3)(4)(-10) = -20 + 120 = 100.$

(b) Use the result of part (a) to show that $A^3 \neq 2A^2$.
**SOLUTION:**
$\det(A^3) = 100^3 = 1000000$, while $\det(2A^2) = 2^4 \cdot 100^2 = 160000$.
So $A^3 \neq 2A^2$ since their determinants are different.

(11) Let $V = P_2$, the vector space of all polynomials of degree at most 2. Let $S = \{1+t, 1-t, t^2\}$ and $T = \{1, 1+t, 1+t^2\}$ be ordered bases for $V$. Let $p(t)$ be a polynomial such that

$$[p(t)]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) Find $p(t)$
**SOLUTION:**
$p(t) = 1 \cdot (1+t) + 2 \cdot (1-t) + 3 \cdot 3 = 3 - t + 3t^2.$

(b) Find the transition matrix $P_{T \leftarrow S}$ between $S$ and $T$.
**SOLUTION:**
The matrix has columns $[1+t]_T, [1-t]_T, [t^2]_T$. We can find these simultaneously using the row-reduction:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Therefore

$$P_{T \leftarrow S} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Find $[p(t)]_T$
**SOLUTION:**

$$[p(t)]_T = P_{T \leftarrow S}[p(t)]_S = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

(12) Let $L : V \to W$ be a linear transformation with $\ker L = \{\mathbf{0}\}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be linearly independent vectors in $V$. Prove that the vectors $L(\mathbf{v}_1), \ldots L(\mathbf{v}_n)$ are also linearly independent.
**SOLUTION:**
Suppose $c_1 L(\mathbf{v}_1) + \ldots + c_n L(\mathbf{v}_n) = \mathbf{0}$. Then $L(c_1 \mathbf{v}_1 + \ldots + c_n \mathbf{v}_n) = \mathbf{0}$.
So $c_1 \mathbf{v}_1 + \ldots + c_n \mathbf{v}_n \in \ker L = \{\mathbf{0}\}$, and so $c_1 \mathbf{v}_1 + \ldots c_n \mathbf{v}_n = \mathbf{0}$.
Since $\mathbf{v}_1, \ldots \mathbf{v}_n$ are linearly independent, this implies $c_1 = \ldots = c_n = 0$.
Thus $L(\mathbf{v}_1), \ldots L(\mathbf{v}_n)$ are linearly independent.

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**MAKEUP EXAM QUESTIONS:**

4(b) Prove that the kernel of $L$ is a subspace of $W$.
**SOLUTION:**
Let $\mathbf{v}_1, \mathbf{v}_2 \in \ker L$. Then $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so $\mathbf{v}_1 + \mathbf{v}_2 \in \ker L$.
Also, if $\mathbf{v} \in \ker L$, $c \in \mathbb{R}$, then $L(c\,\mathbf{v}) = cL(\mathbf{v}) = \mathbf{0}$ so $c\,\mathbf{v} \in \ker L$. Thus the kernel is closed under addition and under scalar multiplication, so it is a subspace.

(5) Let $V$ be an inner product space, and let $\mathbf{u}, \mathbf{v}$ be vectors in $V$. Prove that
$(\mathbf{u}, \mathbf{v}) = \frac{1}{4}\| \mathbf{u} + \mathbf{v} \|^2 - \frac{1}{4}\| \mathbf{u} - \mathbf{v} \|^2$
**SOLUTION:**
$(\mathbf{u}+\mathbf{v})^2 - (\mathbf{u}-\mathbf{v})^2 = (\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}) - (\mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v})$
$= (\mathbf{u}, \mathbf{u}) + (\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{u}) + (\mathbf{v}, \mathbf{v}) - \big((\mathbf{u}, \mathbf{u}) + (\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{u}) + (\mathbf{v}, \mathbf{v})\big)$
$= 2(\mathbf{u}, \mathbf{v}) + 2(\mathbf{v}, \mathbf{u}) = 4(\mathbf{u}, \mathbf{v})$. Dividing by 4, the identity follows.

(12) Let $L : V \to W$ be a linear transformation with $\operatorname{range} L = W$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in $V$, and assume that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a spanning set for $V$.
Prove that $L(\mathbf{v}_1), \ldots L(\mathbf{v}_n)$ are a spanning set for $W$.
**SOLUTION:**
Let $\mathbf{w} \in W$. Then $\mathbf{w} \in \operatorname{range} L$, so $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. Since $\mathbf{v}_1, \ldots \mathbf{v}_n$ are a spanning set for $V$,
$\mathbf{v} = c_1 \mathbf{v}_1 + \ldots c_n \mathbf{v}_n$. So
$\mathbf{w} = L(\mathbf{v}) = L(c_1 \mathbf{v}_1 + \ldots c_n \mathbf{v}_n) = c_1 L(\mathbf{v}_1) + \ldots + c_n L(\mathbf{v}_n)$, so $\mathbf{w} \in \operatorname{span}\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_n)\}$.
Thus $L(\mathbf{v}_1), \ldots, L(\mathbf{v}_n)$ are a spanning set for $W$.