MATH 240 **ALTERNATE FINAL** **5/10/1995**

1. (20 pts) Let A be the following matrix. $A = \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 2 & -2 & -1 & 0 & -1 \\ -3 & 3 & 0 & -3 & -3 \end{bmatrix}$

   (a) Reduce the matrix A to row echelon form.
   (b) Find a basis for the column space of A.
   (c) Find a basis for the nullspace of A.

2. (20 pts) Let A be the following matrix. $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 0 & -4 \\ 0 & 1 & 5 \end{bmatrix}$

   (a) Find the characteristic polynomial of A.
   (b) Find the characteristic values of A. *Hint*: One of the values is $\lambda = 2$.
   (c) Why can the matrix A be diagonalized?

3. (25 pts) Let A be the following matrix. $A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{bmatrix}$

   You are given that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are eigenvectors of A.
   (a) Find the eigenvalues of A that correspond to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.
   (b) Find the inverse of the following matrix. $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$
   (c) For the matrix P in part (b), compute $P^{-1}AP$. *Hint*: Your answer should be a diagonal matrix.

4. (25 pts) Let P_2 be the vector space of all polynomials of degree at most 2. Define the function $L : P_2 \to P_2$ by $L(p(x)) = p(x) + x^2 p''(x)$, for all polynomials $p(x)$ in P_2.
   (a) Show that L is a linear transformation.
   (b) Find the matrix of L relative to the standard basis $S = \{x^2, x, 1\}$.
   (c) Find the rank and nullity of L.

5. (25 pts) Let $\mathbf{q}_1 = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$, $\mathbf{q}_2 = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$, $\mathbf{a}_3 = (1, 1, 1)$.
   (a) Show that $\mathbf{q}_1$ and $\mathbf{q}_2$ are orthogonal, and that each has length 1.
   (b) Use the Gram–Schmidt process to transform the basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{a}_3\}$ into an orthonormal basis.
   Hint: By part (a) you only need to apply the Gram–Schmidt process to the third vector.
   (c) Find the coordinates of $(1, 0, 1)$ relative to the orthonormal basis in part (b).

6. (25 pts) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for the vector space V. Define $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{u}_3 = \mathbf{v}_2 + \mathbf{v}_3$.
   (a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for V.
   (b) Define a linear transformation $L : V \to V$ by letting $L(\mathbf{v}_1) = \mathbf{u}_1$, $L(\mathbf{v}_2) = \mathbf{u}_2$, $L(\mathbf{v}_3) = \mathbf{u}_3$. Choose a basis S for V and find the matrix for L relative to S.
   (c) Find the rank and nullity of the linear transformation L defined in part (b).

7. (30 pts) For each of the following subsets, decide whether or not the subset is a subspace of the given vector space. If it *is* a subspace, show that it satisfies the necessary conditions. If it is *not* a subspace, explain why not.
   (a) $\{(x, y, z) \mid 2x - 3y + 4z = 0\}$ in $\mathbf{R}_3$.
   (b) $\{p(x) \mid p(0) = 2\}$ in the vector space P_2 of all polynomials of degree at most 2.
   (c) The set of all diagonal 2×2 matrices in the vector space M_{22} of all 2×2 matrices.

8. (30 pts) For each of the following questions, write out a careful proof, in complete sentences.
   (a) Let $L_1 : V_1 \to V_2$ and $L_2 : V_2 \to V_3$ be linear transformations. Prove that the composition $L_2 L_1$, defined from V_1 to V_3, is also a linear transformation.
   (b) Prove that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors, then $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.
   (c) Prove that if A and B are similar matrices, then $\det(A) = \det(B)$.