

1. (20 pts) Let  $A$  be the following matrix. 
$$A = \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 2 & -2 & -1 & 0 & -1 \\ -3 & 3 & 0 & -3 & -3 \end{bmatrix}$$

- (a) Reduce the matrix  $A$  to row echelon form.  
 (b) Find a basis for the column space of  $A$ .  
 (c) Find a basis for the nullspace of  $A$ .

2. (20 pts) Let  $A$  be the following matrix. 
$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 0 & -4 \\ 0 & 1 & 5 \end{bmatrix}$$

- (a) Find the characteristic polynomial of  $A$ .  
 (b) Find the eigenvalues of  $A$ .  
 (c) Why can the matrix  $A$  be diagonalized?

3. (25 pts) Let  $A$  be the following matrix. 
$$A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{bmatrix}$$

You are given that  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  are eigenvectors of  $A$ .

- (a) Find the eigenvalues of  $A$  that correspond to  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$ .

- (b) Find the inverse of the following matrix. 
$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

- (c) For the matrix  $P$  in part (b), compute  $P^{-1}AP$ .

4. (25 pts) Let  $P_2$  be the vector space of all polynomials of degree at most 2. Define the function  $L : P_2 \rightarrow P_2$  by  $L(p(x)) = p(x) + x^2p''(x)$ , for all polynomials  $p(x)$  in  $P_2$ .

- (a) Show that  $L$  is a linear transformation.  
 (b) Find the matrix of  $L$  relative to the standard basis  $S = \{x^2, x, 1\}$ .  
 (c) Find the rank and nullity of the matrix you found in part (b).

5. (25 pts) Let  $\mathbf{w}_1 = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$ ,  $\mathbf{w}_2 = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ ,  $\mathbf{u}_3 = (1, 1, 1)$ .

- (a) Show that  $\mathbf{w}_1$  and  $\mathbf{w}_2$  are orthogonal, and that each has length 1.  
 (b) Use the Gram-Schmidt process to transform the basis  $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}_3\}$  into an orthonormal basis.

*Hint:* You only need to apply the Gram-Schmidt process to the third vector.

- (c) Find the coordinates of  $(1, 0, 1)$  relative to the orthonormal basis in part (b).

6. (30 pts) For each of the following subsets, decide whether or not the subset is a subspace of the given vector space. If it *is* a subspace, show that it satisfies the necessary conditions. If it is *not* a subspace, explain why not.

- (a)  $\{(x, y, z) \mid 2x - 3y + 4z = 0\}$  in  $\mathbf{R}_3$ .  
 (b)  $\{p(x) \mid p(0) = 2\}$  in the vector space  $P_2$  of all polynomials of degree at most 2.  
 (c) The set of all diagonal  $2 \times 2$  matrices in the vector space  $M_{22}$  of all  $2 \times 2$  matrices.

7. (10 pts) Define a linear transformation  $L : \mathbf{R}_3 \rightarrow \mathbf{R}_3$  by letting  $L(\mathbf{v})$  be the projection of the vector  $\mathbf{v}$  on the plane  $x + y + z = 0$ . Choose a basis  $S$  for  $\mathbf{R}_3$ , and find the matrix of  $L$  relative to the basis you choose.  
*Hint:* Consider choosing a basis that consists of two vectors in the plane, and one that is perpendicular to the plane.

8. (45 pts) Answer **3** of the following **5** questions.

For each of the questions, write out a careful proof, in complete sentences.

(A) Prove that for any matrix  $A$ , the matrix  $(A^T)A$  is symmetric.

(B) Prove that for any square matrix  $A$ , the nullity of  $A^T$  is equal to the nullity of  $A$ .

(C) Let  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  be a basis for the vector space  $V$ . Define  $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_3$ ,  $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$ ,  $\mathbf{u}_3 = \mathbf{v}_2 + \mathbf{v}_3$ .

Show that  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  is a basis for  $V$ .

(D) Prove that if  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal vectors, then  $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ .

(E) Prove that if  $A$  and  $B$  are similar matrices, then  $\det(A) = \det(B)$ .