MATH 240 QUIZ 4 (Takehome) NAME Solutions

Prof. J. Beachy Due Monday, 10/1/2007 Score / 20

1. (5 pts) Find a set of vectors that spans the null space of the matrix $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 2 & 0 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$.

The first step is to row-reduce the matrix.

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 2 & 0 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -4 & 2 \\ 0 & -2 & -8 & 4 \\ 0 & -1 & -4 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives us the solution $x_1 = 5x_3 - 3x_4$, $x_2 = -4x_3 + 2x_4$. If $x_3 = r$ and $x_4 = s$, then $\begin{bmatrix} 5r - 3x \\ -4r + 2s \\ r \\ s \end{bmatrix} = r \begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ gives a typical solution, and so these two vectors span the null space.

2. (5 pts) Do $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ span $M_{22}$?

Yes, since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{b+c}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{b-c}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

I just found this solution by guessing, which *does* answer the question. For a more systematic solution, if you solve the following system you will get the above answer.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

3. (5 pts) Are the vectors $(1, 1, -1)$, $(0, 1, 1)$, $(1, 1, 1)$, $(1, 2, -2)$ linearly independent? If not, write one vector as a linear combination of the other three.

Solve $x_1(1, 1, -1) + x_2(0, 1, 1) + x_3(1, 1, 1) + x_4(1, 2, -2) = (0, 0, 0)$. This leads to

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ -1 & 1 & 1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The solution is $x_1 = -2x_4$, $x_2 = -x_4$, $x_3 = x_4$. Choose $x_4 = 1$, which gives $(-2)(1, 1, -1) + (-1)(0, 1, 1) + (1)(1, 1, 1) + (1)(1, 2, -2) = (0, 0, 0)$, so $(1, 2, -2) = 2(1, 1, -1) + (0, 1, 1) - (1, 1, 1)$.

4. (5 pts) Let $V$ be the vector space of all continuous functions. Are the "vectors" $f_1(x) = x$, $f_2(x) = e^x$, and $f_3(x) = \sin x$ linearly independent?

Solve $c_1 x + c_2 e^x + c_3 \sin x = 0$. The equation holds for all $x$, so we really have infinitely many equations in 3 unknowns.

The equation you get when you substitute $x = 0$ immediately gives you $c_2 = 0$, so then we only need to solve $c_1 x + c_3 \sin x = 0$.

Now, if you substitute $x = \pi$, you get $\pi c_1 = 0$, and therefore $c_1 = 0$.

Finally, then, if you substitute $x = \pi/2$ you have $c_3 = 0$, and so this proves that the functions are linearly independent.

Another method of solution is to put the 3 equations you get into a matrix and row reduce. Since it reduces to the identity, the functions are linearly independent.

*Comment*: What we have really shown is that there is no "linear" identity that connects these three functions.