MATH 240 QUIZ 7 Solutions 10/19/2007

1. (6 pts) Find this determinant by using elementary row operations to put it in row echelon form.

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 2 & 2 & 3 & 1 \\ -3 & 0 & 2 & 2 \\ 0 & 2 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 2 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{vmatrix} = (1)(2)(-1)(3) = -6$$

2. (8 pts) For the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, verify that $\det(AB) = \det(A) \cdot \det(B)$ by calculating AB, $\det(AB)$, $\det(A)$, and $\det(B)$. We have $\det(A) = 4 - 6 = -2$ and $\det(B) = 4 + 1 = 5$.

$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 10 & 5 \end{bmatrix}$, so $\det(AB) = 20-30 = -10$, which shows that $\det(AB) = \det(A)\cdot\det(B)$.

3. (6 pts) Find $\begin{vmatrix} x-1 & -2 & 2 \\ 0 & x & 1 \\ 0 & -1 & x+2 \end{vmatrix}$ (leave your answer in factored form).

The best solution is probably to evaluate the determinant via cofactors along the first column. There is only one nonzero term, so

$$\begin{vmatrix} x-1 & -2 & 2 \\ 0 & x & 1 \\ 0 & -1 & x+2 \end{vmatrix} = (x-1)\begin{vmatrix} x & 1 \\ -1 & x+2 \end{vmatrix} = (x-1)(x(x+2)+1) = (x-1)(x^2+2x+1) = (x-1)(x+1)^2.$$

Note: We could evaluate the determinant in block form, which gives the same answer $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = |A| \cdot |C|$.

MATH 240 QUIZ 8 Solutions 11/2/2007

1. (10 pts) (the Pythagorean theorem) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.
Prove that $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ if and only if $(\mathbf{u},\mathbf{v}) = 0$.

Use the definition of length to expand both sides of the equation

$||\mathbf{u}+\mathbf{v}||^2 = (\mathbf{u}+\mathbf{v}\,,\mathbf{u}+\mathbf{v}) = (\mathbf{u}\,,\mathbf{u}) + 2(\mathbf{u}\,,\mathbf{v}) + (\mathbf{v}\,,\mathbf{v})$

$||\mathbf{u}||^2 + ||\mathbf{v}||^2 = (\mathbf{u}\,,\mathbf{u}) + (\mathbf{v}\,,\mathbf{v})$

Comparing the two expressions, we can see that the only difference is the term $2(\mathbf{u}\,,\mathbf{v})$ on the left hand side of the equality. The two expressions are equal if and only if $2(\mathbf{u}\,,\mathbf{v}) = 0$, and this occurs if and only if $(\mathbf{u}\,,\mathbf{v}) = 0$.

Note: In $\mathbf{R}^n$, this shows that the Pythagorean theorem holds if and only if you have a right triangle. The same result holds for any inner product defined on any inner product space.

2. (10 pts) Define an inner product on the space P_2 of polynomials of degree ≤ 2 by $(p(x), q(x)) = \int_0^1 p(x)q(x)\,dx$. Using this inner product, find the length of $p_1(x) = x$, and the length of $p_2(x) = x^2$.

$$||x|| = \sqrt{(x,x)} = \sqrt{\int_0^1 x \cdot x\,dx} = \sqrt{\int_0^1 x^2\,dx} = \sqrt{\left.\frac{x^3}{3}\right|_0^1} = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}$$

$$||x^2|| = \sqrt{(x^2,x^2)} = \sqrt{\int_0^1 x^2 \cdot x^2\,dx} = \sqrt{\int_0^1 x^4\,dx} = \sqrt{\left.\frac{x^5}{5}\right|_0^1} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$$