1. (5 pts; p283 #18) Using the notion of the rank of a matrix, determine which of the given matrices are nonsingular.

(a) $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 2 & 3 \\ 0 & 8 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 4 & 1 \\ 0 & 8 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ so the $3 \times 3$ matrix has rank 3 and is nonsingular.

(b) $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 4 & 1 \\ 0 & 0 & 3/4 \end{bmatrix}$ so this matrix is also nonsingular because its rank equals its size.

2. (5 pts) Find this determinant by using elementary row operations to put it in row echelon form.

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 2 & 2 & 3 & 1 \\ -3 & 0 & 2 & 2 \\ 0 & 2 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 2 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{vmatrix} = (1)(2)(-1)(3) = -6$$

3. (5 pts) For the matrices $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, verify that $\det(AB) = \det(A) \cdot \det(B)$ by calculating $AB$, $\det(AB)$, $\det(A)$, and $\det(B)$. We have $\det(A) = 4 - 6 = -2$ and $\det(B) = 4 + 1 = 5$.

$AB = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 0 & 10 \end{bmatrix}$, so $\det(AB) = -10$, which shows that $\det(AB) = \det(A) \cdot \det(B)$.

4. (5 pts) Find $\begin{vmatrix} x-1 & -2 & 2 \\ 0 & x & 1 \\ 0 & -1 & x+2 \end{vmatrix}$ (leave your answer in factored form).

The best solution is probably to evaluate the determinant via cofactors along the first column. There is only one nonzero term, so

$$\begin{vmatrix} x-1 & -2 & 2 \\ 0 & x & 1 \\ 0 & -1 & x+2 \end{vmatrix} = (x-1) \begin{vmatrix} x & 1 \\ -1 & x+2 \end{vmatrix} = (x-1)(x(x+2)+1) = (x-1)(x^2+2x+1) = (x-1)(x+1)^2.$$

*Note:* We could evaluate the determinant in block form, which gives the same answer $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = |A| \cdot |C|$.