

1. (15 pts) Does the system $(a, b, c) = x(2, 1, 3) + y(-1, 2, 0) + z(1, 8, 6)$ have a solution for all values of (a, b, c) ? Hint: To answer the question, row reduce a related matrix.

The corresponding system of linear equations is

$$\begin{aligned} 2x - y + z &= a \\ x + 2y + 8z &= b \\ 3x + 6z &= c \end{aligned}$$

The answer depends only on the coefficient matrix. By conditions (3) and (4) on page 120, we only need to see whether or not the coefficient matrix is row equivalent to the identity.

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 8 \\ 3 & 0 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -3 \\ 0 & 2 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution does not always exist since the coefficient matrix does not reduce to the identity.

2. (10 pts) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & a \end{bmatrix}$, where a is a nonzero real number.

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & 1 \end{array} \right] &\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -3 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & 1 \end{array} \right] &\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 2 & 0 \\ 0 & -1 & 0 & -3 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & 1 \end{array} \right] &\rightsquigarrow \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 2 & 0 \\ 0 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & a & 0 & 0 & 1 \end{array} \right] &\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 2 & 0 \\ 0 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/a \end{array} \right] \end{aligned}$$

3. (10 pts) Write the matrix $\begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$ as a product of elementary matrices.

$$\begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying the same elementary row operations to the identity matrix, we get the following corresponding elementary matrices.

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$$

$$\text{Then } E_3 E_2 E_1 A = I, \text{ so } A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

4. (10 pts) Suppose that A, B are invertible $n \times n$ matrices. Explain why $(AB)^{-1} = A^{-1}B^{-1}$ if and only if $AB = BA$. To give the explanation you need to do the following two cases:

(a) Given that $AB = BA$, show that $(AB)^{-1} = A^{-1}B^{-1}$.

$$(AB)^{-1} = (BA)^{-1} = A^{-1}B^{-1}$$

(b) Given that $(AB)^{-1} = A^{-1}B^{-1}$, show that $AB = BA$.

$$AB = ((AB)^{-1})^{-1} = (A^{-1}B^{-1})^{-1} = (B^{-1})^{-1}(A^{-1})^{-1} = BA$$

5. (30 pts) Determine whether the given subset W is a subspace of the vector space V .

(a) Let $V = \mathbf{R}^3$ and let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x + y + z = 0\}$.

A typical vector has the form $(x, y, z) = (x, y, -x - y) = x(1, 0, -1) + y(0, 1, -1)$. The set of linear combinations of two vectors is a subspace.

Alternate proof: the zero vector has the correct form; W is closed under sums since

$$(x_1, y_1, -x_1 - y_1) + (x_2, y_2, -x_2 - y_2) = (x_1 + x_2, y_1 + y_2, -x_1 - y_1 - x_2 - y_2) = (x_1 + x_2, y_1 + y_2, -(x_1 + y_1) - (x_2 + y_2))$$

and it is closed under scalar multiplication since

$$r \cdot (x, y, -x - y) = (rx, ry, r(-x - y)) = (rx, ry, -(rx + ry)).$$

(b) Let $V = \mathbf{R}^3$ and let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$.

This is not a vector space since $(1, 0, 0)$ is in W but $2 \cdot (1, 0, 0)$ is not in W .

(c) Let V be the vector space P_3 of all polynomials of degree at most 3. Let W be the set of all polynomials $p(x)$ in P_3 for which $\int_0^1 p(x) dx = 0$.

The integral of the zero function is 0; if $p(x)$ and $q(x)$ are polynomials with $\int_0^1 p(x) dx = 0$ and $\int_0^1 q(x) dx = 0$, then $\int_0^1 (p(x) + q(x)) dx = \int_0^1 p(x) dx + \int_0^1 q(x) dx = 0 + 0 = 0$ and $\int_0^1 kp(x) dx = k \int_0^1 p(x) dx = k \cdot 0 = 0$.

6. (15 pts) In \mathbf{R}^2 , use ordinary addition $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, but define a new scalar multiplication by $r \cdot (x, y) = (rx, y)$.

(a) Prove that the distributive law $r \cdot (\mathbf{v}_1 + \mathbf{v}_2) = r \cdot \mathbf{v}_1 + r \cdot \mathbf{v}_2$ holds for all scalars r and all vectors $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$ in \mathbf{R}^2 .

$$r \cdot (\mathbf{v}_1 + \mathbf{v}_2) = r \cdot ((x_1, y_1) + (x_2, y_2)) = r \cdot (x_1 + x_2, y_1 + y_2) = (r(x_1 + x_2), y_1 + y_2)$$

$$r \cdot (x_1, y_1) + r \cdot (x_2, y_2) = (rx_1, y_1) + (rx_2, y_2) = (rx_1 + rx_2, y_1 + y_2)$$

(b) Find one of the laws for scalar multiplication that does *not* hold. State the law and give a numerical example to show that it fails.

$(c + d) \cdot \mathbf{v} = c \cdot \mathbf{v} + d \cdot \mathbf{v}$ fails since $(1 + 1) \cdot (0, 1) = 2 \cdot (0, 1) = (0, 1)$ but $1 \cdot (0, 1) + 1 \cdot (0, 1) = (0, 1) + (0, 1) = (0, 2)$.

7. (10 pts) Let A be any $m \times n$ matrix. Explain why $A^T A$ is a symmetric matrix.

$$(A^T A)^T = A^T (A^T)^T = A^T A$$

The class average was 74.1. The grading scale was
86-98 A (4); 74-84 B (6); 64-73 C (5); 46-57 D (3)