



5. (30 pts) Determine whether the given subset  $W$  is a subspace of the vector space  $V$ . In each part, either check that all three of the necessary conditions hold, or give a numerical counterexample to one of them.

(a) Let  $V = \mathbf{R}^3$  and let  $W = \{(x, y, z) \mid z = 2x - y\}$ .

The set  $W$  is a subspace. The zero vector belongs; if  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  belong, then we have  $z_1 = 2x_1 - y_1$  and  $z_2 = 2x_2 - y_2$ . Their sum is  $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$ , and  $z_1 + z_2 = (2x_1 - y_1) + (2x_2 - y_2) = 2(x_1 + x_2) - (y_1 + y_2)$ , which shows that the sum belongs to  $W$ . For any scalar  $r$ , the product is  $r(x_1, y_1, z_1) = (rx_1, ry_1, rz_1)$ , and  $rz_1 = r(2x_1 - y_1) = 2(rx_1) - ry_1$ , which shows that the scalar product also belongs to  $W$ .

You can also show that  $W$  is a subspace by working with “typical” elements of the form  $(x, y, 2x - y)$ .

(b) Let  $V = \mathbf{R}^2$  and let  $W = \{(x, y) \mid x^2 + y^2 \leq 1\}$ .

This is not a subspace. The vector  $(1, 0)$  belongs to  $W$ , but  $2(1, 0)$  does not.

*Comment:* Geometrically, this region is a disk with center at the origin. It can't be a subspace because it is not one of the possibilities: the entire plane, a line through the origin, or just the origin itself.

(c) Let  $V = \mathbf{R}^n$ , let  $A$  be an  $m \times n$  matrix, and let  $W$  be all vectors  $\mathbf{x}$  in  $V$  with  $A\mathbf{x} = \mathbf{0}$ .

Since  $A\mathbf{0} = \mathbf{0}$ , it follows that  $\mathbf{0}$  is in  $W$ .

If  $\mathbf{x}_1$  and  $\mathbf{x}_2$  belong to  $W$ , then  $A\mathbf{x}_1 = \mathbf{0}$  and  $A\mathbf{x}_2 = \mathbf{0}$ , so  $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$ , and therefore  $\mathbf{x}_1 + \mathbf{x}_2$  is in  $W$ .

Finally, for any scalar  $c$  we have  $A(c \cdot \mathbf{x}_1) = c \cdot A\mathbf{x}_1 = c \cdot \mathbf{0} = \mathbf{0}$ , and so  $c \cdot \mathbf{0}$  is in  $W$ .

6. (15 pts) In  $\mathbf{R}^2$ , use ordinary addition  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ , but define a new scalar multiplication by  $r \cdot (x, y) = (rx, -ry)$ . Check each of the four laws 5, 6, 7, and 8 for scalar multiplication. If the law is valid, give a proof. If not, give a numerical counterexample to show that it fails.

Axiom 5 holds:  $r \cdot ((x_1, y_1) + (x_2, y_2)) = r \cdot (x_1 + x_2, y_1 + y_2) = (r(x_1 + x_2), -r(y_1 + y_2)) = (r(x_1 + x_2), -ry_1 - ry_2)$  and  $r \cdot (x_1, y_1) + r \cdot (x_2, y_2) = (rx_1, -ry_1) + (rx_2, -ry_2) = (rx_1 + rx_2, -ry_1 - ry_2)$ .

Axiom 6 holds:  $(r + s) \cdot (x_1, y_1) = ((r + s)x_1, -(r + s)y_1) = ((r + s)x_1, -ry_1 - sy_1)$  and  $r \cdot (x_1, y_1) + s \cdot (x_1, y_1) = (rx_1, -ry_1) + (sx_1, -sy_1) = (rx_1 + sx_1, -ry_1 - sy_1)$ .

Axiom 7 does not hold since, for example,  $(1 \cdot 2) \cdot (3, 4) = 2 \cdot (3, 4) = (6, -8)$  but  $1 \cdot (2 \cdot (3, 4)) = 1 \cdot (6, -8) = (6, 8)$ .

Axiom 8 does not hold since, for example,  $1 \cdot (0, 1) = (0, -1) \neq (0, 1)$ .