

1. (25 pts) $A = \begin{bmatrix} 1 & -1 & 1 & 5 \\ -2 & 2 & 4 & 2 \\ 2 & -2 & -3 & 0 \\ -3 & 3 & 4 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 0 & 6 & 12 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & 7 & 14 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) The rank of A is 2; its nullity is $4 - 2 = 2$. (c) Basis for the row space: $(1, -1, 0, 3), (0, 0, 1, 2)$.

(d) Basis for the column space of A : columns 1 and 3 of the original matrix

(e) Find a basis for the nullspace of A . The system reduces to this:

$x_1 - x_2 + 3x_4 = 0$ and $x_3 + 2x_4 = 0$ $x_1 = x_2 - 3x_4$ and $x_3 = -2x_4$. Letting $x_2 = 1, x_4 = 0$

gives $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$; letting $x_2 = 0, x_4 = 0$ gives $\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$; these are the 2 basis vectors for the solution space.

2. (10 pts) Determine whether the given sets of vectors are linearly dependent or linearly independent.

(a) $\{(1, 3, -2), (-2, -6, 4)\}$ in \mathbf{R}^3 Dependent: the second is a multiple of the first.

(b) $\{t^3, t^2 - 1, t^2 + 1, 4t, 2t - 3\}$ in P_3 Dependent: there are 5 vectors and $\dim(P_3) = 4$.

3. (10 pts) Find the determinant by reducing to row echelon form.

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 2 & 2 & 3 & 1 \\ -3 & 0 & 2 & 2 \\ 0 & 2 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 2 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{vmatrix} = (1)(2)(-1)(3) = -6$$

4. (10 pts) Solve for x : $\begin{vmatrix} x & 0 & 0 & 1 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 1 & 0 & 0 & x \end{vmatrix} = 0$. $\begin{vmatrix} x & 0 & 0 & 1 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 1 & 0 & 0 & x \end{vmatrix} = x \begin{vmatrix} x & 0 & 1 \\ 0 & x & 0 \\ 1 & 0 & x \end{vmatrix} = x^2 \begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix}$ using

cofactor expansions, so we get $x^2(x^2 - 1) = 0$, and the equation $x^2(x - 1)(x + 1)$ has solutions $x = 0, 1, -1$.

The matrix can also be row reduced by interchanging rows 1 and 4 and then subtracting x times row 1 from row 4. Dividing a row by x causes problems, because you can only do that if x is nonzero, and you have to be careful to give a separate solution in case $x = 0$.

5. (10 pts) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be linearly independent vectors in a vector space V . Explain why the set $\{\mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3\}$ is a linearly independent set.

First solution: set $c_1\mathbf{v}_2 + c_2(\mathbf{v}_3 - \mathbf{v}_2) + c_3(\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3) = 0$.

Then $c_3\mathbf{v}_1 + (c_1 - c_2 + c_3)\mathbf{v}_2 + (c_2 - c_3)\mathbf{v}_3 = 0$, and so $c_3 = 0, c_1 - c_2 + c_3 = 0$, and $c_2 - c_3 = 0$. This leads immediately to the solution $c_3 = 0, c_2 = 0, c_3 = 0$, and this shows that the vectors form a linearly independent set.

Second solution: since $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent, they form a basis for the subspace W that they generate. We can therefore find the coordinate vectors relative to S , put these into a matrix, and row reduce.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \text{ The rank is 3 so the vectors are linearly independent.}$$

7. (10 pts) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be column vectors in \mathbf{R}^3 , and let A be a 3×3 matrix. Prove that if $\{A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3\}$ form a basis for \mathbf{R}^3 , then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ also form a basis for \mathbf{R}^3 .

Solve $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$.

Multiply by A and simplify to get $c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + c_3(A\mathbf{v}_3) = 0$.

Then $c_1 = 0, c_2 = 0, c_3 = 0$ since $\{A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3\}$ are linearly independent.

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent, so they form a basis since $\dim(\mathbf{R}^3) = 3$.

6. (15 pts) Let $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ and $T = \{(1, -1, 0), (0, 1, -1), (0, 0, 1)\}$ be ordered bases for \mathbf{R}_3 . Let $\mathbf{v} = (3, 2, 1)$.

(a) Find the coordinate vector $[\mathbf{v}]_T$ of \mathbf{v} with respect to the basis T .

The equation $(3, 2, 1) = x_1(1, -1, 0) + x_2(0, 1, -1) + x_3(0, 0, 1)$ leads to the equations $x_1 = 3$, $-x_1 + x_2 = 2$, and $-x_2 + x_3 = 1$, with solution $x_1 = 3$, $x_2 = 5$, $x_3 = 6$. The coordinate vector is $[\mathbf{v}]_T = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$.

(b) Find the transition matrix $P_{S \leftarrow T}$. Shorthand: $[S|T] \rightsquigarrow [I|P_{S \leftarrow T}]$.

Note that the coordinates of the vectors in S and T are entered as columns.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

(c) Use $P_{S \leftarrow T}$ to find the coordinate vector $[\mathbf{v}]_S$ of \mathbf{v} with respect to the basis S .

$$P_{S \leftarrow T} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } [\mathbf{v}]_S = P_{S \leftarrow T} [\mathbf{v}]_T = [\mathbf{v}]_T, \text{ so } [\mathbf{v}]_S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Check: $1 \cdot (1, 0, 0) + 1 \cdot (1, 1, 0) + 1 \cdot (1, 1, 1) = (3, 2, 1)$.

8. (10 pts) Recall that a matrix is skew symmetric if $A^T = -A$.

(a) Give an example of a skew symmetric 2×2 matrix whose determinant is *not* equal to zero.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ is skew symmetric with determinant } 1.$$

(b) Explain why $\det(A) = 0$ if A is a skew symmetric $n \times n$ matrix, where n is an *odd* number.

Since $A^T = -A$, taking the determinant gives $\det(A^T) = \det(-A) = (-1)^n \det(A)$, and therefore $\det(A) = -\det(A)$ since n is odd. Then $\det(A) = 0$.

The class average was 76.1. The grading was as follows: 85–92 A (5); 74–83 B (5); 55–72 C (7). Overall, the performance of the class was good. I've posted the grades on blackboard, including the averages on the quiz grades. At the end of the semester, I'll drop the lowest quiz or homework grade.