MATH 240 **Sample Questions** for Exam 3 on 11/30/2007

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1. Let $L : \mathbf{R}^5 \to \mathbf{R}^4$ be the linear transformation defined by $L(\mathbf{x}) = A\mathbf{x}$, for the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 & -1 \\ 1 & 0 & 0 & 2 & -1 \\ 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}. \text{ Given that } A \text{ row-reduces to } \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

(a) find a basis for $\ker L$;

(b) find a basis for $\operatorname{range} L$;

(c) find $\dim(\ker L)$ and $\dim(\operatorname{range} L)$.

2. Define the linear transformation $L : P_2 \to P_2$ by $L(p(t)) = p(t) + 2p'(t)$.

(a) Find the matrix $M_{S \leftarrow S}(L)$ of L relative to the standard basis $S = \{t^2, t, 1\}$.

(b) Find the matrix $M_{T \leftarrow T}(L)$ of L relative to the basis $T = \{t^2 + t + 1, t + 1, 1\}$.

3. Let $L : \mathbf{R}_3 \to \mathbf{R}_3$ be the linear transformation defined by $L(x_1, x_2, x_3) = (x_1,\, x_2 + 2x_3,\, 2x_2 + x_3)$. Let $S = \{(1,0,0),\, (0,1,0),\, (0,0,1)\}$ be the standard basis for $\mathbf{R}_3$, and let T be the basis $\{(1,0,0),\, (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),\, (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$.

(a) Find the matrix $M_{S \leftarrow S}(L)$ of L with respect to the basis S.

(b) Find the matrix $M_{T \leftarrow T}(L)$ of L with respect to the basis T.

(c) Find the transition matrices $P_{S \leftarrow T}$ and $P_{T \leftarrow S}$ that change coordinates.

(d) Check that $M_{T \leftarrow T}(L) = P_{T \leftarrow S} \cdot M_{S \leftarrow S}(L) \cdot P_{S \leftarrow T}$.

4. Show that if the $n \times n$ matrix B is similar to the matrix A, then B^T is similar to A^T.

5. Let W be the subspace of R_4 spanned by the vectors $(1,-1,1,1)$ and $(1,0,2,1)$. Use the Gram-Schmidt process to find an orthonormal basis for W.

6. Let M_{22} be the vector space of all 2×2 matrices. For A, B in M_{22}, define an inner product by $(A, B) = \operatorname{tr}(B^T A)$. (*Recall:* $\operatorname{tr}(A)$ denotes the trace of A, which is the sum of entries on the main diagonal.)

(a) Check that $(A, B) = (B, A)$ for all A, B in M_{22}.

(b) For any 2×2 matrix A, check that $(A, A) = 0$ if and only if $A = 0$.

7. Answer EITHER part **A** OR part **B**.

A. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space V, prove the *parallelogram law*, which states that $||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$.

B. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthogonal set of nonzero vectors in an inner product space V. Show that S is a linearly independent set.

8. Find the orthogonal complement in $\mathbf{R}^4$ of the subspace W spanned by the vectors $(1,1,0,0)$, $(0,1,1,0)$, $(0,0,1,1)$.