Definition of a Vector Space

Definition 4.4. A **vector space** is a set V on which two operations are defined, called **vector addition** and **scalar multiplication**, and denoted by $+$ and $\cdot$, respectively.

The operation $+$ (vector addition) must satisfy the following conditions:

Closure: For all vectors $\mathbf{u}$ and $\mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ belongs to V.

(1) *Commutative law:* For all vectors $\mathbf{u}$ and $\mathbf{v}$ in V, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(2) *Associative law:* For all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

(3) *Additive identity:* The set V contains an **additive identity** element, denoted by $\mathbf{0}$, such that for all vectors $\mathbf{v}$ in V, $\mathbf{0} + \mathbf{v} = \mathbf{v}$ and $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

(4) *Additive inverses:* For each vector $\mathbf{v}$ in V, the equations $\mathbf{v} + \mathbf{x} = \mathbf{0}$ and $\mathbf{x} + \mathbf{v} = \mathbf{0}$ have a solution $\mathbf{x}$ in V, called an **additive inverse** of $\mathbf{v}$, and denoted by $-\mathbf{v}$.

The operation $\cdot$ (scalar multiplication) must satisfy the following conditions:

Closure: For all real numbers c and all vectors $\mathbf{v}$ in V, the product $c \cdot \mathbf{v}$ belongs to V.

(5) *Distributive law:* For all real numbers c and all vectors $\mathbf{u}$, $\mathbf{v}$ in V, $c \cdot (\mathbf{u} + \mathbf{v}) = c \cdot \mathbf{u} + c \cdot \mathbf{v}$.

(6) *Distributive law*: For all real numbers c, d and all vectors $\mathbf{v}$ in V, $(c + d) \cdot \mathbf{v} = c \cdot \mathbf{v} + d \cdot \mathbf{v}$.

(7) *Associative law*: For all real numbers c, d and all vectors $\mathbf{v}$ in V, $c \cdot (d \cdot \mathbf{v}) = (cd) \cdot \mathbf{v}$.

(8) *Unitary law*: For all vectors $\mathbf{v}$ in V, $1 \cdot \mathbf{v} = \mathbf{v}$.

Subspaces

Definition 4.5. Let V be a vector space, and let W be a subset of V. If W is a vector space with respect to the operations in V, then W is called a **subspace** of V.

Theorem 4.3. Let V be a vector space, with operations $+$ and $\cdot$, and let W be a subset of V. Then W is a subspace of V if and only if the following conditions hold.

W is nonempty: The zero vector belongs to W.

Closure under $+$: If $\mathbf{u}$ and $\mathbf{v}$ are any vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.

Closure under $\cdot$: If $\mathbf{v}$ is any vector in W, and c is any real number, then $c \cdot \mathbf{v}$ is in W.