1. (a) (5 pts) Complete this statement of the division algorithm: For any integers $a$ and $b$, with $b > 0$, there exist . . . . .

   (b) (15 pts) Use the division algorithm to prove the part of Theorem 1.1.4 which states that if $I$ is a nonempty set of integers that is closed under addition and subtraction, then $I$ contains a positive integer $b$ such that $b \mid n$ for all $n$ in $I$.

2. (10 pts) Find $\gcd(78, 102)$ and write it as a linear combination of 78 and 102.

3. (a) (5 pts) Complete the following definition: The nonzero integers $a$ and $b$ are said to be relatively prime if . . . . .

   (b) (10 pts) Prove Proposition 1.2.3 (a), which states that if $(a, b, c)$ are integers such that $b \mid ac$, then $b \mid (a, b) \cdot c$.

   (c) (10 pts) Prove or disprove the following statement, for integers $0 < a < b < c$: $\gcd(a + b, c) = 1 \iff \gcd(a - b, c) = 1$.

4. (a) (5 pts) Complete the following statement of part of Theorem 1.3.5:
   The congruence $ax \equiv b \pmod{n}$ has a solution if and only if . . . . .

   (b) (5 pts) Give an example of a linear congruence (of the form $ax \equiv b \pmod{n}$) which has no solution. Explain your answer.

   (c) (10 pts) Find all solutions to the congruence $21x \equiv 6 \pmod{45}$.

5. (a) (5 pts) Explain what is meant by the notation $\mathbb{Z}^\times_n$.

   (b) (5 pts) Complete this statement of Euler’s theorem (which involves powers of congruence classes): If $[a]_n \in \mathbb{Z}^\times_n$, then . . . . .

6. (15 pts) Choose either Part A or Part B.

   Part A

   Let $[a]^n \in \mathbb{Z}^\times_n$. The multiplicative order of $[a]_n$ is the smallest positive integer $k$ such that $[a]^k_n = [1]_n$. Prove that the multiplicative order of $[a]_n$ is a divisor of $\varphi(n)$.

   Part B

   Prove that if $0 < n < m$, then $2^{2^n} + 1$ and $2^{2^m} + 1$ are relatively prime.