MATH 420 | EXAM I | October 2, 2006
Prof. John Beachy | No calculators. | Show all necessary work.

1. (a) [5 pts] Complete the following statement of part of Theorem 1.3.5:

   (i) Let a, b and $n > 1$ be integers. The congruence $ax \equiv b \pmod{n}$ has a solution if and only if ......

   (ii) If there is a solution, then how many distinct solutions are there modulo n?

   (b) [15 pts] Find all solutions to the congruence $91x \equiv 21 \pmod{105}$.

2. (a) [10 pts] Solve the system of congruences

$$\begin{aligned} x &\equiv 5 \pmod{25} \\ x &\equiv 23 \pmod{32}\ . \end{aligned}$$

   (b) [10 pts] Prove that if the system

$$\begin{aligned} x &\equiv 1 \pmod{m} \\ x &\equiv 0 \pmod{n} \end{aligned}$$

   has a solution, then m and n are relatively prime.

3. (a) [10 pts] If a and n are positive integers, explain what is meant by the notations $[a]_n$ and $[a]_n^{-1}$. State necessary and sufficient conditions on a and n which guarantee the existence of $[a]_n^{-1}$.

   (b) [10 pts] Compute $[75]_{112}^{-1}$ (in $\mathbf{Z}_{112}$).

4. (a) [5 pts] Complete this statement of the division algorithm: For any integers a and b, with $b > 0$, there exist ......

   (b) [15 pts] Use the division algorithm to prove the part of Theorem 1.1.4 which states that if I is a nonempty set of integers that is closed under addition and subtraction, then I contains a positive integer b such that $b \mid n$ for all n in I.

5. (a) [10 pts] Prove Proposition 1.2.3 (a), which states that if (a, b, c) are integers such that $b \mid ac$, then $b \mid (a, b) \cdot c$.

   (b) [10 pts] Prove or disprove the following statement, for integers $0 < a < b < c$: $\gcd(a + b, c) = 1 \iff \gcd(b - a, c) = 1$.