1. (a) [5 pts] Complete the following statement of part of Theorem 1.3.5:
   (i) Let $a, b$ and $n > 1$ be integers. The congruence $ax \equiv b \pmod{n}$ has a solution if and only if . . . . . .
   (ii) If there is a solution, then how many distinct solutions are there modulo $n$?

   (b) [15 pts] Find all solutions to the congruence $91x \equiv 21 \pmod{105}$.

2. (a) [10 pts] Solve the system of congruences
   
   \[
   x \equiv 5 \pmod{25} \\
   x \equiv 23 \pmod{32}.
   \]

   (b) [10 pts] Prove that if the system
   
   \[
   x \equiv 1 \pmod{m} \\
   x \equiv 0 \pmod{n}
   \]
   
   has a solution, then $m$ and $n$ are relatively prime.

3. (a) [10 pts] If $a$ and $n$ are positive integers, explain what is meant by the notations $[a]_n$ and $[a]_n^{-1}$. State necessary and sufficient conditions on $a$ and $n$ which guarantee the existence of $[a]_n^{-1}$.

   (b) [10 pts] Compute $[75]_{112}^{-1}$ (in $\mathbb{Z}_{112}$).

4. (a) [5 pts] Complete this statement of the division algorithm: For any integers $a$ and $b$, with $b > 0$, there exist . . . . . .

   (b) [15 pts] Use the division algorithm to prove the part of Theorem 1.1.4 which states that if $I$ is a nonempty set of integers that is closed under addition and subtraction, then $I$ contains a positive integer $b$ such that $b \mid n$ for all $n$ in $I$.

5. (a) [10 pts] Prove Proposition 1.2.3 (a), which states that if $(a, b, c)$ are integers such that $b \mid ac$, then $b \mid (a, b) \cdot c$.

   (b) [10 pts] Prove or disprove the following statement, for integers $0 < a < b < c$:
   \[
   \gcd(a + b, c) = 1 \iff \gcd(b - a, c) = 1.
   \]