MATH 420 TEST I September 29, 1995
Prof. John Beachy

20 points per problem. Show all necessary work.

1. (a) Find all solutions to the congruence $55x \equiv 35 \pmod{75}$.
   (b) Find all solutions to the congruence $55x \equiv 36 \pmod{75}$.

2. (a) Solve the system of congruences

$$\begin{aligned} x &\equiv 5 \pmod{25} \\ x &\equiv 23 \pmod{32}\ . \end{aligned}$$

   (b) Give integers a, b, m, n to provide an example of a system

$$\begin{aligned} x &\equiv a \pmod{m} \\ x &\equiv b \pmod{n} \end{aligned}$$

   that has no solution.

3. (a) Find the multiplicative inverse of each nonzero element of $\mathbf{Z}_7$.
   (b) Find the multiplicative inverse of each nonzero element of $\mathbf{Z}_{13}$.

4. Prove the result from the text which states that if a, b, c are any integers, then $\gcd(a, bc) = 1$ if and only if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$.

5. Let p be a prime number. Prove that

$$(p-1)! \equiv -1 \pmod{p}\ .$$

   (You can get partial credit for verifying the result for $p = 7$ and $p = 13$.)

MATH 420 TEST I February 14, 2003
Prof. John Beachy Show all necessary work. No calculators.

1. (a) [5 pts] Complete the following statement of part of Theorem 1.3.5:
   The congruence $ax \equiv b \pmod{n}$ has a solution if and only if ......

   (b) [15 pts] Find all solutions to the congruence $42x \equiv 24 \pmod{72}$.

2. (a) [10 pts] If a and n are positive integers, explain what is meant by the notations $[a]_n$ and $[a]_n^{-1}$. State necessary and sufficient conditions on a and n which guarantee the existence of $[a]_n^{-1}$.

   (b) [10 pts] Compute $[38]_{83}^{-1}$ (in $\mathbf{Z}_{83}$).

3. (a) [10 pts] Give the description of the least common multiple $\mathrm{lcm}[a, b]$ of two nonzero integers a and b in terms of the prime factorizations of a and b.

   (b) [10 pts] Explain why this fact is true: if p is a prime number such that $p \mid \mathrm{lcm}[a, b]$, then either $p \mid a$ or $p \mid b$.

4. (a) [5 pts] State the division algorithm.

   (b) [15 pts] Prove the part of Theorem 1.1.4 which states that if I is a nonempty set of integers that is closed under addition and subtraction, then I contains a positive integer b such that $b \mid n$ for all n in I.

5. (a) [15 pts] Let p and q be distinct prime numbers. Prove that

$$p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}\ .$$

   Hint: Recall Fermat's theorem, which states that if p is prime then $a^p \equiv a \pmod{p}$ for any integer a.

   (b) [5 pts] Compute $5^6 + 7^4 \pmod{35}$. *Note:* If you first prove part (a), you will automatically get full credit for (b), without doing any computations.