MATH 420 TEST II October 27, 2006
Prof. John Beachy

1. (15 pts) Define $f : \mathbf{Z}_8 \to \mathbf{Z}_{12}$ by $f([x]_8) = [3x]_{12}$, for all $[x]_8 \in \mathbf{Z}_8$.

   (a) Show that f is a well-defined function.

   Recall: you must show that if $x_1 \equiv x_2 \pmod 8$, then $3x_1 \equiv 3x_2 \pmod{12}$.

   (b) Find the image $f(\mathbf{Z}_8)$ and the set of equivalence classes $\mathbf{Z}_8/f$ defined by f, and exhibit the one-to-one correspondence between these sets.

2. (25 pts) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 1 & 8 & 3 & 6 & 4 & 7 & 9 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 5 & 4 & 7 & 2 & 6 & 8 & 9 & 3 \end{pmatrix}$.

   (a) Write each of σ, τ, $\sigma\tau$, $\tau\sigma$, and $\sigma\tau\sigma^{-1}$ as a product of disjoint cycles.

   (b) Find the order of each of σ, τ, $\sigma\tau$, $\tau\sigma$, and $\sigma\tau\sigma^{-1}$.

   Recall: the order of a permutation σ is the smallest positive exponent m for which σ^m is equal to the identity.

   (c) Determine whether each of σ, τ, $\sigma\tau$, $\tau\sigma$, and $\sigma\tau\sigma^{-1}$ is an even permutation or an odd permutation.

3. (20 pts) Let $f : S \to T$ and $g : T \to U$ be functions.

   (a) State these definitions: f is **one-to-one**; f is **onto**.

   (b) Prove that if gf is a one-to-one function, then so is f.

   (c) Prove that if gf is an onto function, then so is g.

4. (20 pts) For integers m, n, b with $n > 1$, define $f : \mathbf{Z}_n \to \mathbf{Z}_n$ by $f([x]_n) = [mx + b]_n$. *You may assume that f is a well-defined function.*

   Prove that f is a one-to-one correspondence if and only if $\gcd(m, n) = 1$. Then find the inverse function f^{-1}, assuming that $\gcd(m, n) = 1$.

5. (10 pts) Let S be the set of all $n \times n$ matrices with real entries. For $A, B \in S$, define $A \sim B$ if there exists an invertible matrix P such that $B = PAP^{-1}$. Prove that $\sim$ is an equivalence relation.

6. (10 pts) Let $\sigma \in S_n$ have order m. Prove that if k is any integer, then $\sigma^k = (1)$ if and only if $m \mid k$.

   The rules: You must give a direct proof that does not use Proposition 2.3.7 from the text, which states that $\sigma^i = \sigma^j$ if and only if $i \equiv j \pmod m$.