MATH 420 TEST III December 4, 2006

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Show all of the work necessary to justify your answers. You may not use a calculator.

Each problem is worth 20 points.

1. (a) State the definition of a **group**.

   (b) State the definition of an **abelian** group.

   (c) Give an example of a finite group that is not abelian. Explain your answer.

2. (a) For each $\sigma \in S_3$, find $\langle \sigma \rangle$, the cyclic subgroup generated by σ.

   (b) Find the order of each element of the group $\mathbf{Z}_4 \times \mathbf{Z}_4^\times$.

3. (a) What are the possibilities for the order of an element of $\mathbf{Z}_{11}^\times$? Explain your answer.

   (b) Show that $\mathbf{Z}_{11}^\times$ is a cyclic group.

4. (a) In the group $G = GL_2(\mathbf{R})$ of invertible 2×2 matrices with real entries, show that

$$H = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in GL_2(\mathbf{R}) \mid a_{11} = 1 \,, a_{21} = 0 \,, a_{22} = 1 \right\}$$

   is a subgroup of G.

   (b) Show that H is isomorphic to the group $\mathbf{R}$ of all real numbers, under addition.

5. *Choose Part A **OR** Part B.*

   Part A. Prove Proposition 3.4.5: Assume that m and n are positive integers such that $\gcd(m, n) = 1$. For $k = mn$, define $\phi : \mathbf{Z}_k \to \mathbf{Z}_m \times \mathbf{Z}_n$ by $\phi([x]_k) = ([x]_m, [x]_n)$, for all $[x]_k \in \mathbf{Z}_k$. Prove that ϕ is a well-defined function, and that ϕ is an isomorphism.

   Part B. State and prove Lagrange's theorem.

   In your proof you may assume Lemma 3.2.9, which states that if H is a subgroup of a group G, and for $a, b \in G$ we define $a \sim b$ if $ab^{-1} \in H$, then $\sim$ is an equivalence relation on G.