1. (c) Give an example of a finite group that is not abelian. Explain your answer.

S_3 is the smallest nonabelian group. To show it is nonabelian, compute:
$(1,2)(2,3) = (1,2,3)$ but $(2,3)(1,2) = (3,2)(2,1) = (3,2,1)$.

2. (a) For each $\sigma \in S_3$, find $\langle \sigma \rangle$, the cyclic subgroup generated by σ. $\langle (1,2,3) \rangle = \{(1),(1,2,3),(1,3,2)\} = \langle (1,3,2) \rangle$

$\langle (1,2) \rangle = \{(1),(1,2)\}$ $\langle (1,3) \rangle = \{(1),(1,3)\}$ $\langle (2,3) \rangle = \{(1),(2,3)\}$ $\langle (1) \rangle = \{(1)\}$

2. (b) Find the order of each element of the group $\mathbf{Z}_4 \times \mathbf{Z}_4^\times$.

$\mathbf{Z}_4 = \{0,1,2,3\}$ under addition, and $\mathbf{Z}_4^\times = \{1,3\}$, under multiplication, and so $\mathbf{Z}_4 \times \mathbf{Z}_4^\times$ has 8 elements. Use the result that the order of an element in $G_1 \times G_2$ is the least common multiple of the orders of its components.
In $\mathbf{Z}_4$, 0 has order 1, 2 has order 2, and $1,3$ have order 4. In $\mathbf{Z}_4^\times$, 1 has order 1, and 3 has order 2.
Order 1: $(0,1)$ since $\text{lcm}[1,1] = 1$
Order 2: $(2,1), (0,2), (2,2)$ since $\text{lcm}[2,1] = 2$, $\text{lcm}[1,2] = 2$, $\text{lcm}[2,2] = 2$
Order 4: $(1,1), (1,3), (3,1), (3,3)$ since $\text{lcm}[4,1] = 4$, $\text{lcm}[4,2] = 4$, $\text{lcm}[4,1] = 4$, $\text{lcm}[4,2] = 4$

3. (a) What are the possibilities for the order of an element of $\mathbf{Z}_{11}^\times$? Explain your answer.

The group $\mathbf{Z}_{11}^\times$ has order 10. As a consequence of Lagrange's theorem, the order of any element of $\mathbf{Z}_{11}^\times$ must be a divisor of 10. The possible orders are 1, 2, 5, and 10.

3. (b) Show that $\mathbf{Z}_{11}^\times$ is a cyclic group.

The first element to try is 2, and we have $2^2 = 4$, and $2^5 = 32 \equiv 10$, so the order of 2 is greater than 5. By part (a) it must be 10, and thus 2 is a generator for $\mathbf{Z}_{11}^\times$. We can also write this as $\mathbf{Z}_{11}^\times = \langle [2]_{11} \rangle$.

4. (a) In the group $G = GL_2(\mathbf{R})$ of invertible 2×2 matrices with real entries, show that H is a subgroup of G.

$$H = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in GL_2(\mathbf{R}) \mid a_{11} = 1 , a_{21} = 0 , a_{22} = 1 \right\}$$

Closure: $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}$, for all $a, b \in \mathbf{R}$.
Identity: The identity matrix has the correct form to belong to H.
Inverses: $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \in H$, for all $a \in \mathbf{R}$.

Note: it isn't enough to say that each element of H has an inverse. Of course this is true, since G is a group. The point is to show that any element in H has an inverse *in* H.

4. (b) Show that H is isomorphic to the group $\mathbf{R}$ of all real numbers, under addition.

Define $\phi : \mathbf{R} \to H$ by $\phi(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, for all $x \in \mathbf{R}$. If $\phi(x_1) = \phi(x_2)$, then $x_1 = x_2$ since matrices are equal if and only if corresponding entries are equal. It is clear that ϕ is onto, since x can be any element of $\mathbf{R}$. Finally, ϕ respects the operations in H and $\mathbf{R}$ since $\phi(x)\phi(y) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix} = \phi(x+y)$.

Note: You could prove that ϕ is one-to-one and onto by observing that ϕ has an inverse $\theta : H \to \mathbf{R}$ defined by $\theta\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) = x$.

Grades: 78-94 A (6); 62-77 B (2); 46-61 C (9); 35-45 D (3); 12-34 F (4)