3.4.15 Let $G$ be any group, and let $a$ be a fixed element of $G$. Define a function $\phi_a : G \to G$ by $\phi_a(x) = axa^{-1}$, for all $x \in G$. Show that $\phi_a$ is an isomorphism.

3.4.16 Let $G$ be any group. Define $\phi : G \to G$ by $\phi(x) = x^{-1}$, for all $x \in G$.
   (a) Prove that $\phi$ is one-to-one and onto.
   (b) Prove that $\phi$ is an isomorphism if and only if $G$ is abelian.

3.4.22 Let $a, b$ be positive integers, and let $d = \gcd(a, b)$ and $m = \lcm[a, b]$. Write $d = sa + tb$, $a = a'd$, and $b = b'd$. Prove that the function $f : \mathbb{Z}_m \times \mathbb{Z}_d \to \mathbb{Z}_a \times \mathbb{Z}_b$ defined by $f(([x]_m, [y]_d)) = ([x+ysa']_a, [x-ytb']_b)$ is an isomorphism.

3.4.55 (from the Study Guide) Is $\mathbb{Z}_{24}^\times$ isomorphic to $\mathbb{Z}_{30}^\times$? Give a complete explanation for your answer.

3.5.11 Which of the multiplicative groups $\mathbb{Z}_7^\times$, $\mathbb{Z}_{10}^\times$, $\mathbb{Z}_{12}^\times$, $\mathbb{Z}_{14}^\times$ are isomorphic to each other?

3.5.12 Let $a, b$ be positive integers, and let $d = \gcd(a, b)$ and $m = \lcm[a, b]$. Use Proposition 3.5.5 to prove that $\mathbb{Z}_m \times \mathbb{Z}_d \cong \mathbb{Z}_a \times \mathbb{Z}_b$.

3.5.18 Prove that $\sum_{d|n} \varphi(d) = n$ for any positive integer $n$.
   Hint: Interpret the equation in the cyclic group $\mathbb{Z}_n$, by considering all of its subgroups.

3.5.39 (from the Study Guide) Let $G$ be a group with a subgroup $H$, and let $a \in G$ be an element of order $n$. Prove that if $a^m \in H$, where $\gcd(m, n) = 1$, then $a \in H$. 