Homework 2 Solutions

40. Find the prime factorizations of 13651 and 3179 and use them to find gcd(13651, 3179).

I first found the greatest common divisor via the Euclidean algorithm, then factored it to get $187 = 11 \cdot 17$. Then $3179 = 11 \cdot 17^2$ and $13652 = 11 \cdot 17 \cdot 73$.

45. Let $a, b, c$ be positive integers.
   (a) Prove that if gcd$(a, bc) = 1$ and gcd$(b, c) = 1$, then gcd$(ab, c) = 1$.
   Assume that gcd$(a, bc) = 1$. Then Proposition 1.2.3 (d) implies that gcd$(a, c) = 1$. Since gcd$(b, c) = 1$, Proposition 1.2.3 (d) implies that gcd$(ab, c) = 1$.
   (b) Prove or disprove the following generalization of part (a): if gcd$(b, c) = 1$, then gcd$(a, bc) = gcd(ab, c)$.
   Taking $a = 2$, $b = 2$, and $c = 3$ gives a counterexample, since gcd$(a, bc) = 1$ but gcd$(ab, c) = 1$.

46. Let $a, b, c$ be positive integers with $a^2 + b^2 = c^2$.
   (a) Show that gcd$(a, b) = 1$ if and only if gcd$(a, c) = 1$.
   (b) Does gcd$(a, b) = gcd(a, c)$?
   Part (b) is true, and proving it will give us part (a).
   Let $d = gcd(a, b)$. Then $a = dq_1$ and $b = dq_2$ for some $q_1, q_2 \in \mathbb{Z}$, so $c^2 = d^2(q_1^2 + q_2^2)$. It follows from Exercise 17 that $q_1^2 + q_2^2$ is a perfect square, say $c^2 = d'^2 q^2$, so that $c = dq$. This shows that $d \mid c$, so since we already have $d \mid a$ it follows that $d \mid gcd(a, c)$.
   On the other hand, since $a^2 = c^2 - b^2$, a similar argument shows that gcd$(a, c) \mid gcd(a, b)$. If two positive integers are each a factor of the other then they must be equal, so we can conclude that gcd$(a, b) = gcd(a, c)$.