MATH 420 TEST II November 1, 1995

Prof. John Beachy Show all of the work necessary to justify your answers. You may not use a calculator.

1. (15 pts) In S_{10}, let $\alpha = (1,3,5,7,9)$, $\beta = (1,2,6)$, and $\gamma = (1,2,5,3)$. For $\sigma = \alpha\beta\gamma$, write σ as a product of disjoint cycles, and use this to find its order and its inverse. Is σ even or odd?

   Solution: We have $\sigma = (1,6,3,2,7,9)$, so σ has order 6, and $\sigma^{-1} = (1,9,7,2,3,6)$. Since σ has length 6, it can be written as a product of 5 transpositions, so it is an odd permutation.

2. (10 pts) Define the formula $f : \mathbf{Z}_{12} \to \mathbf{Z}_{12}$ by $f([x]_{12}) = [x]_{12}^2$, for all $[x]_{12} \in \mathbf{Z}_{12}$. Show that the formula f defines a function. Find the image of f and the set $\mathbf{Z}_{12}/f$ of equivalence classes determined by f.

   Solution: The formula for f is well-defined since if $[x_1]_{12} = [x_2]_{12}$, then $x_1 \equiv x_2 \pmod{12}$, and so $x_1^2 \equiv x_2^2 \pmod{12}$, which shows that $f([x_1]_{12}) = f([x_2]_{12})$.

   To compute the images of f we have $[0]_{12}^2 = [0]_{12}$, $[\pm 1]_{12}^2 = [1]_{12}$, $[\pm 2]_{12}^2 = [4]_{12}$, $[\pm 3]_{12}^2 = [9]_{12}$, $[\pm 4]_{12}^2 = [4]_{12}$, $[\pm 5]_{12}^2 = [1]_{12}$, and $[6]_{12}^2 = [0]_{12}$. Thus $f(\mathbf{Z}_{12}) = \{[0]_{12}, [1]_{12}, [4]_{12}, [9]_{12}\}$. The corresponding equivalence classes determined by f are $\{[0]_{12}, [6]_{12}\}$, $\{[\pm 1]_{12}, [\pm 5]_{12}\}$, $\{[\pm 2]_{12}, [\pm 4]_{12}\}$, $\{[\pm 3]_{12}\}$.

3. (10 pts) Let $f : S \to T$ be a function. Complete these definitions:

   (a) The function f is one-to-one if ......

   (b) The function f is onto if ......

4. (15 pts) Let $f : S \to T$ and $g : T \to U$ be functions, and assume that f is onto. Show that the composition $g \circ f$ is onto if and only if g is onto.

   Solution: If g is onto, then given any $u \in U$ $\exists$ $t \in T$ with $g(t) = u$, and since f is onto $\exists$ $s \in S$ with $f(s) = t$. Thus $u = g(t) = g(f(s)) = g \circ f(s)$.

   If $g \circ f$ is onto, and $u \in U$, then $\exists$ $s \in S$ with $g \circ f(s) = u$. Thus $u = g(f(s))$, and so g is onto.

5. (15 pts) If A and B are $n \times n$ matrices, we say that B is similar to A, written $B \sim A$, if there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. Show that this relation $\sim$ is an equivalence relation on the set of all $n \times n$ matrices. (That is, check the reflexive, symmetric, and transitive laws.)

   Solution: For any A, we have $I^{-1}AI = A$. If $A \sim B$, then $\exists$ P with $B = P^{-1}AP$, so $A = PBP^{-1} = (P^{-1})^{-1}BP^{-1}$, and thus $B \sim A$. If $A \sim B$ and $B \sim C$, then $\exists$ P, Q with $B = P^{-1}AP$ and $C = Q^{-1}BQ$. A substitution gives $C = Q^{-1}(P^{-1}AP)Q = (PQ)^{-1}A(PQ)$, so $A \sim C$.

6. (15 pts) Let $\sigma \in S_n$, and assume that σ has order m, where $m > 1$. Prove the result from the text which states that for positive integers i, j we have $\sigma^i = \sigma^j$ if and only if $i \equiv j \pmod{m}$.

7. (10 pts) State the definition of a group.

8. (10 pts) Let $G = \{x \in \mathbf{R} \mid x > 1\}$ be the set of all real numbers greater than 1. For $x, y \in G$, define $x * y = xy - x - y + 2$.

   (a) Show that 2 is the identity element for the operation $*$.

   Solution: Since the operation is commutative, the one computation $2 * y = 2y - 2 - y + 2 = y$ suffices to show that 2 is the identity.

   (b) Show that for element $a \in G$ there exists an inverse $a^{-1} \in G$.

   Solution: Given any $a \in G$, we need to solve $a * y = 2$. This gives us the equation $ay - a - y + 2 = 2$, which has the solution $y = a/(a-1)$. This solution belongs to G since $a > a - 1$ implies $a/(a-1) > 1$. Finally, $a * (a/a-1) = a^2/(a-1) - a - a/(a-1) + 2 = (a^2 - a^2 + a - a)/(a-1) + 2 = 2$.

80–91, A (4) 65–78, B (4) 50–54, C (4) 35–43, D (6) 16–29, F (3)