Homework 2 Solutions

40. Find the prime factorizations of 13651 and 3179 and use them to find $\gcd(13651, 3179)$.

I first found the greatest common divisor via the Euclidean algorithm, then factored it to get $187 = 11 \cdot 17$. Then $3179 = 11 \cdot 17^2$ and $13652 = 11 \cdot 17 \cdot 73$.

45. Let a, b, c be positive integers.
 (a) Prove that if $\gcd(a, bc) = 1$ and $\gcd(b, c) = 1$, then $\gcd(ab, c) = 1$.

Assume that $\gcd(a, bc) = 1$. Then Proposition 1.2.3 (d) implies that $\gcd(a, c) = 1$. Since $\gcd(b, c) = 1$, Proposition 1.2.3 (d) implies that $\gcd(ab, c) = 1$.

 (b) Prove or disprove the following generalization of part (a): if $\gcd(b, c) = 1$, then $\gcd(a, bc) = \gcd(ab, c)$.

Taking $a = 2$, $b = 2$, and $c = 3$ gives a counterexample, since $\gcd(a, bc) = 1$ but $\gcd(ab, c) = 1$.

46. Let a, b, c be positive integers with $a^2 + b^2 = c^2$.
 (a) Show that $\gcd(a, b) = 1$ if and only if $\gcd(a, c) = 1$.
 (b) Does $\gcd(a, b) = \gcd(a, c)$?

Part (b) is true, and proving it will give us part (a).

Let $d = \gcd(a, b)$. Then $a = dq_1$ and $b = dq_2$ for some $q_1, q_2 \in \mathbf{Z}$, so $c^2 = d^2(q_1^2 + q_2^2)$. It follows from Exercise 17 that $q_1^2 + q_2^2$ is a perfect square, say $c^2 = d^2q^2$, so that $c = dq$. This shows that $d \mid c$, so since we already have $d \mid a$ it follows that $d \mid \gcd(a, c)$.

On the other hand, since $a^2 = c^2 - b^2$, a similar argument shows that $\gcd(a, c) \mid \gcd(a, b)$. If two positive integers are each a factor of the other then they must be equal, so we can conclude that $\gcd(a, b) = \gcd(a, c)$.