Homework 9

Hand in:

From the study guide:

Section §3.2 #54, 55, 56, 60, 62 (2 pts each)
Note: See Exercise 19 for the definition of the centralizer of an element.

54. In $G = \text{GL}_2(\mathbb{R})$, find the centralizers $C \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)$ and $C \left( \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right)$.

55. Let $G$ be a group, and let $a, b \in G$. Show that if either $ab \in C(a)$ or $ba \in C(a)$, then $b \in C(a)$.

56. Let $G$ be an abelian group, and let $G_n = \{ x \in G \mid x^n = e \}$.
   (a) Show that $G_n$ is a subgroup of $G$.
   
   Note: This is a generalization of Exercise 3.2.13.
   
   (b) Let $G = \mathbb{Z}_{11}$. Find $G_n$ for $n = 2, 3, \ldots, 10$.

60. In $\text{GL}_2(\mathbb{R})$, let $A = \left[ \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]$, which has infinite order, and let $B = \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right]$. Find the order of $B$, and find the order of $AB$.

62. Let $A = \{ f_{m,b} \mid m \neq 0 \text{ and } f_{m,b}(x) = mx + b \text{ for all } x \in \mathbb{R} \}$ shown to be a group in Exercise 3.1.10.
   (a) Show that $\{ f_{1,n} \mid n \in \mathbb{Z} \}$ is a cyclic subgroup of $A$.
   
   (b) Find the cyclic subgroup $\langle f_{2,1} \rangle$ of $A$ generated by the mapping $f_{2,1}(x) = 2x + 1$.

From the text:

Section §3.1 #8, 9, 14, 19 (5 pts each)

8. Let $G = \text{GL}_2(\mathbb{R})$. For each of the following subsets of $M_2(\mathbb{R})$, determine whether or not the subset is a subgroup of $G$.
   
   (a) $A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \left| \begin{array}{c} \text{ab} \neq 0 \end{array} \right. \right\}$
   
   (b) $B = \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \left| \begin{array}{c} bc \neq 0 \end{array} \right. \right\}$
   
   (c) $C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \left| \begin{array}{c} c \neq 0 \end{array} \right. \right\}$

9. Let $G = \text{GL}_3(\mathbb{R})$. Show that $H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right\}$ is a subgroup of $G$.

14. Let $G$ be an abelian group. Show that the set of all elements of $G$ of finite order forms a subgroup of $G$.

19. Let $G$ be a group, and let $a \in G$. The set $C(a) = \{ x \in G \mid xa = ax \}$ of all elements of $G$ that commute with $a$ is called the \textbf{centralizer} of $a$.
   
   (a) Show that $C(a)$ is a subgroup of $G$.
   
   (b) Show that $\langle a \rangle \subseteq C(a)$.
   
   (c) Compute $C(a)$ if $G = S_3$ and $a = (1, 2, 3)$.
   
   (d) Compute $C(a)$ if $G = S_3$ and $a = (1, 2)$. 