Homework #11 Solutions
Assignment due 4/18/2003

Section 3.3, #12 (page 110)
Let G be a group of order 6. Show that G must contain an element of order 2 (see Exercise 21 of Section 3.1). Show that it cannot be true that every element different from e has order 2.
Hint: Show that if every element had order 2 it would be possible to construct a subgroup of order 4.
Solution: If G is any group and $x \in G$ with $x^2 \neq e$, then $x \neq x^{-1}$ and $(x^{-1})^2 = (x^2)^{-1} \neq e$. Thus G has an even number of elements x with $x^2 \neq e$. If the total number of elements in G is even, this leaves an even number of elements x with $x^2 = e$. There is at least one such element, the identity e. Thus there must be at least one more element a, with $a \neq e$ and $a^2 = e$.

Now suppose that $|G| = 6$ and every element of G has order 2. Let $a, b \in G$, $a \neq b, a \neq e, b \neq e$. Then $a^2 = b^2 = e$. Consider $ab \in G$. Since $a \neq b$ and $a^2 = e$, we must have $ab \neq e$. Thus $o(ab) = 2$ and $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$. (You can also show this by quoting Exercise 20 of Section 3.1, which implies that G must be abelian.) By simply computing all of the possible products of elements, it is possible to see that $H = \{e, a, b, ab\}$ is closed under multiplication, so it is a subgroup of G by Corollary 3.2.4. Thus H is a subgroup of order 4, contradicting Lagrange's theorem.

Section 3.3, #13 (page 110)
Let G be a group of order 6, and suppose that $a, b \in G$ with a of order 3 and b of order 2. Show that either G is cyclic or $ab \neq ba$.
Solution: Let $a, b \in G$, with $o(a) = 3$ and $o(b) = 2$. Then for the powers of ab we have $(ab)^2 = a^2b^2 = a^2 \neq e$; $(ab)^3 = a^3b^3 = b \neq e$; $(ab)^4 = a^4b^4 = a \neq e$; $(ab)^5 = a^5b^5 = a^2b \neq e$; $(ab)^6 = a^6b^6 = (a^3)^2(b^2)^3 = e$. (This also follows from Exercise 21 of Section 3.2, which says that if $ab = ba$ and $o(a)$ is relatively prime to $o(b)$, then $o(ab) = \text{lcm}[o(a), o(b)]$.) Since ab has order 6, we have shown that $G = \langle ab \rangle$, and thus G is a cyclic group. If $ab \neq ba$ then G is not abelian and therefore not cyclic.

Section 3.3, #14 (page 110)
Let G be any group of order 6. Show that if G is not cyclic, then its multiplication table must look like that of S_3.
Hint: If the group is not cyclic, use Exercises 12 and 13 to produce elements $a, b \in G$ with $a^3 = e$, $b^2 = e$ and $ba = a^2b$.
Solution: By Exercises 12 and 13, if G is not cyclic then there exists an element a of order 3 and an element b of order 2 such that $ab \neq ba$. Now the elements e, a, a^2, b are all distinct. By cancellation $ab \neq a, a^2, b$ and since $a^{-1} = a^2 \neq b$ we have $ab \neq e$. Thus e, a, a^2, b, ab are all distinct. Again by cancellation $a^2b \neq a, a^2, ab, b$. Also $a^2 \neq b$, so $a^2b \neq e$. Thus $G = \{e, a, a^2, b, ab, a^2b\}$.

What is ba? By cancellation $ba \neq a, a^2, b$. Since $b \neq a^2, ba \neq e$. By assumption $ba \neq ab$. By elimination we can see that we must have $ba = a^2b$. Using this identity, we can compute the remaining products, and so we can complete the multiplication table for G.

$\cdot$	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^b	a^2b	ab	b	a^2	a	e

This is the table on page 104 of the text, and so we have shown that any nonabelian group of order 6 must have a multiplication table that follows the pattern of the table for S_3.