MATH 420 EXAM I Solutions October 2, 2006

Prof. John Beachy

1. (b) [15 pts] Find all solutions to the congruence $91x \equiv 21 \pmod{105}$.

$91 = 7 \cdot 13$ and $105 = 3 \cdot 5 \cdot 7$, so $\gcd(91, 105) = 7$

$91x = 21 + 105q$ so $13x = 3 + 15q$, which gives $-2x \equiv 3 \pmod{15}$. Then $[-2]_{15}^{-1} = [7]$, so $x \equiv 21 \equiv 6 \pmod{15}$, so the solution is $x \equiv 6, 21, 36, 51, 66, 81, 96 \pmod{15}$.

2. (a) [10 pts] Solve the system of congruences

$$\begin{aligned} x &\equiv 5 \pmod{25} \\ x &\equiv 23 \pmod{32}\ . \end{aligned}$$

$x = 23 + 32q$ from the second equation, so substituting into the first equation gives us $23 + 32q \equiv 5 \pmod{25}$, or $7q \equiv 7 \pmod{25}$. We get $q \equiv 1 \pmod{25}$, so the solution is $x \equiv 55 \pmod{25 \cdot 32}$.

(b) [10 pts] Prove that if the system

$$\begin{aligned} x &\equiv 1 \pmod{m} \\ x &\equiv 0 \pmod{n} \end{aligned}$$

has a solution, then m and n are relatively prime.

$x = 1 + mq$ and $x = np$ for some $p, q \in \mathbf{Z}$, and setting these equal we have $1 + mq = np$, so $m(-q) + np = 1$, showing that $\gcd(m, n) = 1$.

3. (b) [10 pts] Compute $[75]_{112}^{-1}$ (in $\mathbf{Z}_{112}$).

$$\begin{bmatrix} 1 & 0 & 112 \\ 0 & 1 & 75 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 37 \\ 0 & 1 & 75 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 37 \\ -2 & 3 & 3 \end{bmatrix}$$

We have $(-2)(112) + 3(75) = 1$, so $3 \cdot 75 \equiv 1 \pmod{112}$, and thus $[75]_{112}^{-1} = [3]_{112}$.

5. (b) [10 pts] Prove or disprove the following statement, for integers $0 < a < b < c$: $\gcd(a + b, c) = 1 \iff \gcd(b - a, c) = 1$.

Let $a = 1$, $b = 2$, and $c = 3$. Then $\gcd(1 + 2, 3) = 3$ but $\gcd(2 - 1, 3) = 1$, so the statement is false.