2. (10 pts) Find $\gcd(323, 391)$ and write it as a linear combination of 323 and 391.

\[
\begin{bmatrix}
1 & 0 & 391 \\
0 & 1 & 323
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 68 \\
-4 & 5 & 51
\end{bmatrix}
\rightarrow
\begin{bmatrix}
5 & -6 & 17 \\
-19 & 23 & 0
\end{bmatrix}
\]

3. (10 pts) Find all integers $x$ such that $5x + 3$ is divisible by 12. What is the smallest positive solution to this problem?

Solve the congruence $5x + 3 \equiv 0 \pmod{12}$. You get $5x \equiv -3 \pmod{12}$, and then since the inverse of 5 is 5, multiplying by 5 gives $x \equiv -15 \pmod{12}$, so $x \equiv 9 \pmod{12}$. The smallest positive solution is $x = 9$.

4. (20 pts)

(a) Solve the system of congruences

\[
\begin{align*}
x & \equiv 5 \pmod{25} \\
x & \equiv 23 \pmod{32}
\end{align*}
\]

Write $x = 23 + 32q$ for some $q \in \mathbb{Z}$, and substitute to get $23 + 32q \equiv 5 \pmod{25}$, which reduces to $7q \equiv 7 \pmod{25}$, so $q \equiv 1 \pmod{15}$. This gives $x \equiv 55 \pmod{25 \cdot 32}$.

(b) Give integers $a, b, m, n$ to provide an example of a system

\[
\begin{align*}
x & \equiv a \pmod{m} \\
x & \equiv b \pmod{n}
\end{align*}
\]

that has no solution.

In the example the integers $m$ and $n$ cannot be relatively prime. This is the clue to take $m = n = 2$, with $a = 1$ and $b = 0$.

7. (10 pts) Prove this fact: for any positive integer $a$ there exists a prime number $p$ such that $a < p \leq a! + 1$.

The number $a! + 1$ has some prime factor $p$. But if $p \leq a$, then $p$ is a factor of $a!$, which would also force it to be a factor of 1. Thus $p > a$, and, of course, $p \leq a! + 1$ since $p$ is a factor of $a! + 1$.

Comment: This is similar to the proof that there are infinitely many prime numbers.