MATH 420 | TEST IV | August 6, 2009
Prof. John Beachy | | 75 minutes

1. (10 pts) (a) State the definition of a group isomorphism.

   (b) Let $G$ be a group, and let $a$ be an element of $G$. Define $\phi : G \to G$ by setting $\phi(x) = axa^{-1}$, for all $x \in G$. Show that $\phi$ is an isomorphism.

2. (20 pts) (a) Prove that if $\phi : G_1 \to G_2$ is a group isomorphism and $H$ is a subgroup of $G_1$, then $\phi(H)$ is a subgroup of $G_2$.

   (Recall that $\phi(H) = \{y \in G_2 \mid y = \phi(x) \text{ for some } x \in H\}$.)

   (b) Prove that if $\phi : G_1 \to G_2$ is a group isomorphism, then $\phi^{-1} : G_2 \to G_1$ is also an isomorphism. (This is Proposition 3.4.2 (a).)

3. (10 pts) Let $G$ be a group, and let $a, b$ be elements of $G$.

   (a) Define $\lambda_a : G \to G$ by setting $\lambda_a(x) = ax$, for all $x \in G$. Show that $\lambda_a$ is a one-to-one and onto function.

   (b) Show that $\lambda_a \circ \lambda_b = \lambda_{ab}$ and that $(\lambda_a)^{-1} = \lambda_{a^{-1}}$.

   (The above questions are part of the proof of Cayley's theorem.)

4. (10 pts) Let $G$ be an abelian group, and let $H$ be the subset of $G$ consisting of all elements of $G$ that have finite order.

   (a) Prove that $H$ is a subgroup of $G$.

   (b) If $G = \mathbf{R}^\times$, what is $H$? If $G = \mathbf{R}$, what is $H$?

5. (20 pts) Let $G$ be a group.

   (a) Let $a \in G$ be an element of order $n$. Prove that if $m \in \mathbf{Z}$, then $\langle a^m \rangle = \langle a^d \rangle$, where $d = \gcd(m, n)$, and $a^m$ has order $n/d$. (This is Proposition 3.5.3.)

   (b) If $a \in G$ has order 48, list all powers of $a$ that have order 12.

6. (10 pts) (a) Let $G$ be a group of order 4, with identity element $e$ and elements $a, b, c$. If $a, b, c$ each have order 2, write out the group table for $G$. Explain why there is only one possible group table.

   (b) Explain why any group of order 4 is isomorphic to either $\mathbf{Z}_4$ or $\mathbf{Z}_2 \times \mathbf{Z}_2$.

7. (10 pts) List the elements of $\mathbf{Z}_{30}^\times$. Is $\mathbf{Z}_{30}^\times$ a cyclic group?

8. (10 pts) Let $n = pq$, where $p$ and $q$ are different odd primes. Prove that $\mathbf{Z}_n^\times$ is not a cyclic group.