MATH 421 FINAL EXAM NAME ______________________

J. Beachy 5/5/2004 Show the work necessary to justify your answers.

Each question is worth 40 points. For each question, answer part (a) on this sheet, and answer parts (b),(c) in your blue book. Note the choice in questions 1(b) and 4(b).

1. (a) Complete the following definitions.

Defn 3.7.1: Let G_1 and G_2 be groups, and let $\phi : G_1 \to G_2$ be a function. Then ϕ is said to be a *group homomorphism* if

Defn 3.7.3: Let $\phi : G_1 \to G_2$ be a group homomorphism. Then the *kernel* of ϕ is the set

Defn 3.7.5: A subgroup H of the group G is called a *normal* subgroup of G if

1. (b) Answer either part I, part II, OR part III.

I. State and prove Cayley's theorem.

OR II. Prove that if G is a cyclic group, then either G is isomorphic to $\mathbf{Z}$ or G is isomorphic to $\mathbf{Z}_n$, where $n = |G|$.

OR III. State and prove the Fundamental Homomorphism Theorem.

2. (a) State the definitions of the following groups.

Defn 1.4.9: The group of units $\mathbf{Z}_n^\times$

Defn 3.3.6: The general linear group $\mathrm{GL}_n(F)$, where F is a field

Defn 3.6.3: The dihedral group D_n, for $n \geq 3$

2. (b) Let $G = \mathbf{Z}_{17}^\times$. Show that G is a cyclic group. Show that the subset $N = \{\pm 1, \pm 4\}$ is a subgroup of G. List the cosets of N in G. To answer the question of whether or not G/N is a cyclic group, prove that any factor group of a cyclic group is cyclic.

3. (a) Complete the statements of the following theorems.

Thrm 4.2.1: (The division algorithm) Let F be a field, and let $f(x)$ and $g(x)$ be polynomials with coefficients in F such that $g(x) \neq 0$. Then

Thrm 4.3.6: (Eisenstein's criterion) Let $f(x) = a_n x^n + \ldots + a_0 \in \mathbf{Z}[x]$. Then $f(x)$ is irreducible over $\mathbf{Q}$ if

Thrm 4.4.8: (Kronecker) Let F be a field, and let $f(x)$ be any nonconstant polynomial in $F[x]$. Then

3. (b) Use Eisenstein's criterion to show that the following polynomial is irreducible over the field $\mathbf{Q}$ of rational numbers: $x^4 - 18x^3 + 30x^2 + 12$

3. (c) Show that $\mathbf{Z}_3[x]/\langle x^3 + x^2 - x + 1 \rangle$ is a field. Find the inverse, in this field, of the congruence class $[x^2 + 1]$.

4. (a) Complete the following definitions.

Defn 5.1.6: A commutative ring R is called an *integral domain* if

Defn 5.3.3: Let R be a commutative ring with identity, and let I be an ideal of R. Then I is called a *principal* ideal of R if

Defn 5.3.3: Let R be a commutative ring with identity. Then R is called a *principal ideal domain* if

4. (b) Answer either part I OR part II.

I. Prove that if F is a field, then the ring of polynomials $F[x]$ with coefficients in F is a principal ideal domain. (*Note:* This is essentially Thrm 4.2.2. You must give a proof from first principles, using the division algorithm, and the fact that the degree of a product is the sum of the degrees of the factors.)

OR II. Prove that every nonzero prime ideal of a principal ideal domain is maximal.

5. (a) Complete the following definitions.

Defn 6.1.2: Let F be an extension field of K, and let $u \in F$. Then u is said to be *algebraic* over K if

Defn 6.2.2: Let F be an extension field of K. Then F is called a *finite* extension of K if

Defn 6.2.8: Let F be an extension field of K. Then F is said to be a *algebraic* over K if

5. (b) Let F be a finite extension field of K. Prove that F is algebraic over K.

5. (c) Let $K \subseteq E \subseteq F$ be fields. Prove that if E is an algebraic extension of K and F is an algebraic extension of E, then F is an algebraic extension of K.