MATH 421 TEST I February 25, 2004

Prof. John Beachy Show all of the work necessary to justify your answers.

1. (a) (15 pts) State the definitions of the following terms: *group homomorphism*; *normal subgroup*; *kernel of a group homomorphism*.

   (b) (10 pts) Prove that the kernel of any group homomorphism is a normal subgroup.

2. (20 pts) Answer either part I, part II, OR part III.

   I. State and prove Cayley's theorem.

   OR II. Prove that if G is a cyclic group, then either G is isomorphic to $\mathbf{Z}$ or G is isomorphic to $\mathbf{Z}_n$, where $n = |G|$.

   OR III. State and prove the Fundamental Homomorphism Theorem.

3. Assume that the dihedral group of order 16 is described as all elements of the form

$$D_8 = \{a^i b^j \mid 0 \le i < 8,\ 0 \le j < 2\}\ ,$$

   subject to the conditions that $o(a) = 8$, $o(b) = 2$, and $ba = a^{-1}b$, and that the dihedral group of order 8 is described as all elements of the form

$$D_4 = \{c^i d^j \mid 0 \le i < 4,\ 0 \le j < 2\}\ ,$$

   subject to the conditions that $o(c) = 4$, $o(d) = 2$, and $dc = c^{-1}d$.

   Define the mapping $\phi : D_8 \to D_4$ by $\phi(a^i b^j) = c^{2i} d^j$.

   (a) (5 pts) Show that ϕ is a group homomorphism.

   (b) (10 pts) Find the image of ϕ and the kernel of ϕ.

   (c) (5 pts) Exhibit the one-to-one correspondence between the elements of $\text{Im}(\phi)$ and the cosets of $\ker(\phi)$ that is determined by ϕ.

4. Let $G = \mathbf{Z}_{19}^{\times}$ (the multiplicative group of nonzero congruence classes modulo 19).

   (a) (5 pts) Find the cyclic subgroup N generated by the congruence class of 7.

   (b) (5 pts) List the cosets of N.

   (c) (5 pts) Why is N a normal subgroup of G?

   (d) (10 pts) Show that the factor group G/N is cyclic.

5. (10 pts) Let G be a group, and let $H \subseteq N$ be subgroups of G. Prove or disprove the following statement:

   If N is a normal subgroup of G and H is a normal subgroup of N, then H is a normal subgroup of G.

   (Either give a proof of the statement, or show that there is a specific group G with subgroups H and N for which the statement does not hold.)

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(3) Assume that the dihedral group of order 16 is described as all elements of the form

$$D_8 = \{a^i b^j \mid 0 \le i < 8,\ 0 \le i < 2\}\ ,$$

subject to the conditions that $o(a) = n$, $o(b) = 2$, and $ba = a^{-1}b$, and that the dihedral group of order 8 is described as all elements of the form

$$D_4 = \{c^i d^j \mid 0 \le i < 4,\ 0 \le i < 2\}\ ,$$

subject to the conditions that $o(c) = n$, $o(d) = 2$, and $dc = c^{-1}d$. Define $\phi : D_8 \to D_4$ by $\phi(a^i b^j) = c^{2i} d^j$.

(a) (5 pts) Show that ϕ is a group homomorphism.

$\phi(a^ib^ja^sb^t) = \phi(a^{i-s}b^{j+t}) = c^{2(i-s)}d^{j+t} = c^{2i-2s}d^{j+t}$
$\phi(a^ib^j)\phi(a^sb^t) = c^{2i}d^jc^{2s}d^t = c^{2i-2s}d^{j+t}$ These are equal for all $0 \le i < 8$ and $0 \le i < 2$.

(b) (10 pts) Find the image of ϕ and the kernel of ϕ. $\text{Im}(\phi) = \{e, c^2, d, c^2d\}$ $\ker(\phi) = \{e, a^2, a^4, a^6\}$

(c) Exhibit the one-to-one correspondence between the elements of $\text{Im}(\phi)$ and the cosets of $K = \ker(\phi)$ that is determined by ϕ.

$K = \{e, a^2, a^4, a^6\} \longleftrightarrow e$ $aK = \{a, a^3, a^5, a^7\} \longleftrightarrow c^2$
$bK = \{b, a^2b, a^4b, a^6b\} \longleftrightarrow d$ $abK = \{ab, a^3b, a^5b, a^7b\} \longleftrightarrow c^2d$

4. Let $G = \mathbf{Z}_{19}^{\times}$ (the multiplicative group of nonzero congruence classes modulo 19).
(a) (5 pts) Find the cyclic subgroup N generated by the congruence class of 7.
$\langle 7 \rangle = \{1, 7, 11\}$, since $7^2 = 49 \equiv 11 \pmod{19}$, and $7^3 \equiv 77 \equiv 1 \pmod{19}$.
(b) (5 pts) List the cosets of N.
$\langle 7 \rangle$ $2\langle 7 \rangle = \{2, 14, 3\}$ $4\langle 7 \rangle = \{4, 9, 6\}$ $5\langle 7 \rangle = \{5, 16, 17\}$ $8\langle 7 \rangle = \{8, 18, 12\}$ $10\langle 7 \rangle = \{10, 13, 15\}$
(c) (5 pts) Why is N a normal subgroup of G? Any subgroup of an abelian group is normal.
(d) (10 pts) Show that the factor group G/N is cyclic. The coset $2\langle 7 \rangle$ is a generator:
$4\langle 7 \rangle = 2^2\langle 7 \rangle$ $8\langle 7 \rangle = 2^3\langle 7 \rangle$ $5\langle 7 \rangle = 16\langle 7 \rangle = 2^4\langle 7 \rangle$ $10\langle 7 \rangle = 13\langle 7 \rangle = 2^5\langle 7 \rangle$

5. (10 pts) Let G be a group, and let $H \subseteq N$ be subgroups of G. Prove or disprove the following statement:
If N is a normal subgroup of G and H is a normal subgroup of N, then H is a normal subgroup of G.

The statement is false. For a counterexample, recall the homework problem on factor groups of D_4.

Let $G = D_4$, let $N = \{e, a^2, b, a^2b\}$, and let $H = \{e, b\}$. The set N is nonempty and closed, so it is a subgroup. Since it has half the number of elements of G it is normal in G. The set H is the subgroup $\langle b \rangle$, and it is a normal subgroup of N since N is abelian (any group with 4 elements is abelian). Finally, H is not a normal subgroup of G since $aH = \{a, ab\}$ while $Ha = \{a, a^3b\}$.