MATH 421 TEST I February 21, 1996

Prof. John Beachy Show all of the work necessary to justify your answers. You may not use a calculator.

1. (a) (15 pts) State the definitions of the following terms: *group homomorphism*; *normal subgroup*; *kernel of a group homomorphism*.

   (b) (5 pts) State a condition that is equivalent to the definition of a normal subgroup (and can be used to check whether or not a subgroup is normal).

   (c) (5 pts) Give an example of a group homomorphism that is not an isomorphism.

2. Let G be the group $\mathbf{Z}_3 \times \mathbf{Z}_6$ (remember that the group operation is componentwise addition), and let N be the subgroup $\langle ([2]_3, [2]_6) \rangle$ generated by the element $([2]_3, [2]_6)$.

   (a) (10 pts) Find the cosets of N.

   (b) (10 pts) Find the order of each coset of N in the factor group G/N.

3. Assume that the dihedral group of order $2n$ is given as all elements of the form

$$D_n = \{a^i, a^i b \mid 0 \le i < n\} ,$$

   subject to the conditions that $o(a) = n$, $o(b) = 2$, and $ba = a^{-1}b$. Let N be any subgroup of D_n with $N \subseteq \langle a \rangle$, where $\langle a \rangle$ is the cyclic subgroup generated by a.

   (a) (15 pts) Show that N is normal in D_n.

   (b) (10 pts) Show that if $n > 2$ and a^2 is not in N, then the factor group D_n/N is not abelian. (You may use the homework problem which states that a factor group G/N is abelian if and only if $xyx^{-1}y^{-1} \in N$, for all $x, y \in G$.)

4. (20 pts) Answer either part I or part II.

   I. State and prove Cayley's theorem.

   OR

   II. State and prove the Fundamental Homomorphism Theorem.

5. Let G be the group $\mathbf{Q}$ of all rational numbers, under addition, and let N be the subgroup $\mathbf{Z}$ of all integers.

   (a) (5 pts) Show that for each positive integer n there is a subgroup of G/N isomorphic to $\mathbf{Z}_n$.

   (b) (5 pts) Show that G/N has no subgroup isomorphic to $\mathbf{Z}$.