Each question is worth 25 points. Note the choice in questions 7 and 8.

1. State the definitions of normal subgroup and left coset. Show that if a subgroup $N$ of a group $G$ is normal, then multiplication of left cosets is well-defined.

2. State the Fundamental Homomorphism Theorem for groups, and use it to prove that any cyclic group is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_n$, for some $n$.

3. Assume that the dihedral group $D_4$ is given as \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}, where $a^4 = e$, $b^2 = e$, and $ba = a^3b$. Let $N$ be the set \{e, a^2\}. Show by a direct computation that $N$ is a normal subgroup of $D_4$. Is the factor group $D_4/N$ a cyclic group?

4. Construct a field with 8 elements.

5. Let $R$ be a commutative ring with identity 1. For any element $a \in R$, show that $\text{Ann}(a) = \{r \in R \mid ra = 0\}$ is an ideal of $R$. Find $\text{Ann}(a)$ for the element $a = (0, 2)$ of the ring $R = \mathbb{Z}_{12} \oplus \mathbb{Z}_8$.

6. State the definitions of finite extension field, algebraic element, and algebraic extension field. Prove that any finite extension field is an algebraic extension.

7. Choose either A or B.

   A. Let $K \subseteq E \subseteq F$ be fields. Prove that if $E$ is an algebraic extension of $K$ and $F$ is an algebraic extension of $E$, then $F$ is an algebraic extension of $K$.

   B. Show that $\mathbb{Q}(\sqrt{2} + i) = \mathbb{Q}(\sqrt{2}, i)$. Find the minimal polynomial of $\sqrt{2} + i$ over $\mathbb{Q}$.

8. Choose either A or B.

   A. Prove that if $F$ is a field, then the ring of polynomials $F[x]$ with coefficients in $F$ is a principal ideal domain.

   B. Prove that any nonzero prime ideal of a principal ideal domain is maximal.