FINAL EXAM, MATH 421

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Each question is worth 25 points. Note the choice in questions 7 and 8.

1. State the definitions of *normal subgroup* and *left coset*. Show that if a subgroup N of a group G is normal, then multiplication of left cosets is well-defined.

2. State the Fundamental Homomorphism Theorem for groups, and use it to prove that any cyclic group is isomorphic to either $\mathbf{Z}$ or $\mathbf{Z}_n$, for some n.

3. Assume that the dihedral group D_4 is given as $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, where $a^4 = e$, $b^2 = e$, and $ba = a^3b$. Let N be the set $\{e, a^2\}$. Show by a direct computation that N is a normal subgroup of D_4. Is the factor group D_4/N a cyclic group?

4. Construct a field with 8 elements.

5. Let R be a commutative ring with identity 1. For any element $a \in R$, show that $\text{Ann}(a) = \{r \in R \mid ra = 0\}$ is an ideal of R. Find $\text{Ann}(a)$ for the element $a = (0, 2)$ of the ring $R = \mathbf{Z}_{12} \oplus \mathbf{Z}_8$.

6. State the definitions of *finite extension field*, *algebraic element*, and *algebraic extension field*. Prove that any finite extension field is an algebraic extension.

7. Choose either A or B.

   A. Let $K \subseteq E \subseteq F$ be fields. Prove that if E is an algebraic extension of K and F is an algebraic extension of E, then F is an algebraic extension of K.

   B. Show that $\mathbf{Q}(\sqrt{2} + i) = \mathbf{Q}(\sqrt{2}, i)$. Find the minimal polynomial of $\sqrt{2} + i$ over $\mathbf{Q}$.

8. Choose either A or B.

   A. Prove that if F is a field, then the ring of polynomials $F[x]$ with coefficients in F is a principal ideal domain.

   B. Prove that any nonzero prime ideal of a principal ideal domain is maximal.