1. Use the Euclidean algorithm to find \( \gcd(x^8 - 1, x^6 - 1) \) in \( \mathbb{Q}[x] \) and write it as a linear combination of \( x^8 - 1 \) and \( x^6 - 1 \).

2. Are the following polynomials irreducible over \( \mathbb{Q} \)?
   (a) \( 3x^5 + 18x^2 + 24x + 6 \)
   (b) \( 7x^3 + 12x^2 + 3x + 45 \)
   (c) \( 2x^{10} + 25x^3 + 10x^2 - 30 \)

3. (a) Show that \( x^2 + 1 \) is irreducible over \( \mathbb{Z}_3 \).
   (b) List the elements of the field \( F = \mathbb{Z}_3[x]/\langle x^2 + 1 \rangle \).
   (c) In the multiplicative group of nonzero elements of \( F \), show that \([x + 1]\) is a generator, but \([x]\) is not.

4. In \( \mathbb{Z}_2[x]/\langle x^3 + x + 1 \rangle \), find the multiplicative inverse of \([x + 1]\).

5. In \( \mathbb{Z}_5[x]/\langle x^3 + x + 1 \rangle \), find \([x]^{-1}\) and \([x + 1]^{-1}\), and use your answers to find \([x^2 + x]^{-1}\).

6. Let \( R \) be the ring with 8 elements consisting of all \( 3 \times 3 \) matrices with entries in \( \mathbb{Z}_2 \) which have the following form:
   \[
   \begin{bmatrix}
   a & 0 & 0 \\
   0 & a & 0 \\
   b & c & a
   \end{bmatrix}
   \]
   You may assume that the standard laws for addition and multiplication of matrices are valid.
   (a) Show that \( R \) is a commutative ring (you only need to check closure and commutativity of multiplication).
   (b) Find all units of \( R \), and all nilpotent elements of \( R \).
   (c) Find all idempotent elements of \( R \).

7. Let \( R \) be the ring \( \mathbb{Z}_2[x]/\langle x^2 + 1 \rangle \). Show that although \( R \) has 4 elements, it is not isomorphic to either of the rings \( \mathbb{Z}_4 \) or \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

8. In the group \( \mathbb{Z}_{180}^* \) of units of the ring \( \mathbb{Z}_{180} \), what is the largest possible order of an element?

9. For the element \( a = (0,2) \) of the ring \( R = \mathbb{Z}_{12} \oplus \mathbb{Z}_8 \), find \( \text{Ann}(a) = \{ r \in R \mid ra = 0 \} \). Show that \( \text{Ann}(a) \) is an ideal of \( R \).

10. Let \( I \) be the subset of \( \mathbb{Z}[x] \) consisting of all polynomials with even coefficients. Prove that \( I \) is a prime ideal; prove that \( I \) is not maximal.

11. Let \( R \) be the ring \( \mathbb{Z}_2[x]/\langle x^3 + 1 \rangle \).
    (a) Find all ideals of \( R \).
    (b) Find the units of \( R \).
    (c) Find the idempotent elements of \( R \).

12. Let \( S \) be the ring \( \mathbb{Z}_2[x]/\langle x^3 + x \rangle \).
    (a) Find all ideals of \( S \).
    (b) Find the units of \( R \).
    (c) Find the idempotent elements of \( R \).

13. Show that the rings \( R \) and \( S \) in the two previous problems are isomorphic as abelian groups, but not as rings.

14. Let \( I \) and \( J \) be ideals in the commutative ring \( R \), and define the function \( \phi : R \to R/I \oplus R/J \) by \( \phi(r) = (r + I, r + J) \), for all \( r \in R \).
    (a) Show that \( \phi \) is a ring homomorphism, with \( \ker(\phi) = I \cap J \).
    (b) Show that if \( I + J = R \), then \( \phi \) is onto, and thus \( R/(I \cap J) \cong R/I \oplus R/J \).