1. Use the Euclidean algorithm to find $\gcd(x^8 - 1,\ x^6 - 1)$ in $\mathbf{Q}[x]$ and write it as a linear combination of $x^8 - 1$ and $x^6 - 1$.

2. Are the following polynomials irreducible over $\mathbf{Q}$?

   (a) $3x^5 + 18x^2 + 24x + 6$

   (b) $7x^3 + 12x^2 + 3x + 45$

   (c) $2x^{10} + 25x^3 + 10x^2 - 30$

3. (a) Show that $x^2 + 1$ is irreducible over $\mathbf{Z}_3$.

   (b) List the elements of the field $F = \mathbf{Z}_3[x]/\left\langle x^2 + 1 \right\rangle$.

   (c) In the multiplicative group of nonzero elements of F, show that $[x + 1]$ is a generator, but $[x]$ is not.

4. In $\mathbf{Z}_2[x]/\left\langle x^3 + x + 1 \right\rangle$, find the multiplicative inverse of $[x + 1]$.

5. In $\mathbf{Z}_5[x]/\left\langle x^3 + x + 1 \right\rangle$, find $[x]^{-1}$ and $[x + 1]^{-1}$, and use your answers to find $[x^2 + x]^{-1}$.

6. Let R be the ring with 8 elements consisting of all 3×3 matrices with entries in $\mathbf{Z}_2$ which have the following form:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}$$

   You may assume that the standard laws for addition and multiplication of matrices are valid.

   (a) Show that R is a commutative ring (you only need to check closure and commutativity of multiplication).

   (b) Find all units of R, and all nilpotent elements of R.

   (c) Find all idempotent elements of R.

7. Let R be the ring $\mathbf{Z}_2[x]/\left\langle x^2 + 1 \right\rangle$. Show that although R has 4 elements, it is not isomorphic to either of the rings $\mathbf{Z}_4$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

8. In the group $\mathbf{Z}_{180}^{\times}$ of units of the ring $\mathbf{Z}_{180}$, what is the largest possible order of an element?

9. For the element $a = (0, 2)$ of the ring $R = \mathbf{Z}_{12} \oplus \mathbf{Z}_8$, find $\text{Ann}(a) = \{r \in R \mid ra = 0\}$. Show that $\text{Ann}(a)$ is an ideal of R.

10. Let I be the subset of $\mathbf{Z}[x]$ consisting of all polynomials with even coefficients. Prove that I is a prime ideal; prove that I is not maximal.

11. Let R be the ring $\mathbf{Z}_2[x]/\left\langle x^3 + 1 \right\rangle$.

    (a) Find all ideals of R.

    (b) Find the units of R.

    (c) Find the idempotent elements of R.

12. Let S be the ring $\mathbf{Z}_2[x]/\left\langle x^3 + x \right\rangle$.

    (a) Find all ideals of S.

    (b) Find the units of R.

    (c) Find the idempotent elements of R.

13. Show that the rings R and S in the two previous problems are isomorphic as abelian groups, but not as rings.

14. Let I and J be ideals in the commutative ring R, and define the function $\phi : R \to R/I \oplus R/J$ by $\phi(r) = (r + I, r + J)$, for all $r \in R$.

    (a) Show that ϕ is a ring homomorphism, with $\ker(\phi) = I \cap J$.

    (b) Show that if $I + J = R$, then ϕ is onto, and thus $R/(I \cap J) \cong R/I \oplus R/J$.