MATH 421 TEST I October 2, 2009
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1. (25 pts) Given a group G, let $Z(G)$ denote its center.

   (a) Show that $Z(G)$ is a subgroup of G; show that $Z(G)$ is a normal in G.

   We have $e \in Z(G)$ since $eg = g = ge$ for all $g \in G$. Let $a, b \in Z(G)$. Then $(ab)g = a(bg) = a(gb) = (ag)b = (ga)b = g(ab)$ and since $bg = gb$ we can multiply on the left and on the right by b^{-1}, to get $gb^{-1} = b^{-1}g$, for all $g \in G$, so $Z(G)$ is a subgroup. For any $g \in G$ we have $gag^{-1} = agg^{-1} = g \in Z(G)$, so $Z(G)$ is normal.

   (b,c) Show that if n is odd, then $Z(D_n)$ is trivial, and if n is even, then $Z(D_n)$ is not trivial.

   First, the centralizer of a is not all of D_n since it does not include b, so since $\langle a \rangle$ has order n, this must be the centralizer of a. Thus any element in the center is a power of a, which must commute with b. We have $a^i b = ba^i = a^{-i}b$, so $i \equiv -i \pmod{n}$, or $2i \equiv 0 \pmod{n}$. If n is odd, then $\gcd(2, n) = 1$, so we can cancel the 2 to get $i \equiv 0 \pmod{n}$, and thus only $a^0 = e$ is in the center. On the other hand, if n is even, then in the above equation we have the solution $i = n/2$, and so in this case $a^{n/2}$ is in the center.

2. (25 pts) State and prove the fundamental homomorphism theorem for groups. See Theorem 3.8.9.

3. (20 pts) Let N be a normal subgroup of the group G.

   (a) Prove that if G is abelian, then so is G/N. For all $a, b \in G$, $(aN)(bN) = abN = baN = (bN)(aN)$.

   (b) Prove that if G is cyclic, then so is G/N. If $G = \langle a \rangle$, then for all $gN \in G/N$ we have $gN = a^i N = (aN)^i$ for some i, so $G/N = \langle aN \rangle$.

   (c) Give an example of a group G with a normal subgroup N that illustrates that G/N may be cyclic even though G is not cyclic. The group S_3 is not cyclic since it is not abelian, but $\langle a \rangle$ has half the number of elements of S_3, so it is normal, and then $S_3 / \langle a \rangle$ is cyclic since it only has two elements.

4. (30 pts) Solve **3** of the following 5 problems.

   (a) Find all group homomorphisms from $\mathbf{Z}_5^\times$ into $\mathbf{Z}_8^\times$. In each case, find the kernel of the homomorphism.
   We have $\mathbf{Z}_5^\times = \langle 2 \rangle$ since $2^2 = 4 \neq 1$, and therefore 2 has order 4. Every homomorphism is completely determined by what it does to the generator 2, and the image of 2 must be an element whose order is a divisor of 4. Mapping 2 to 1 gives the trivial homomorphism, whose kernel is all of $\mathbf{Z}_5^\times$. Mapping 2 to 3, 5, or 7 is legal since each has order 2, and in each of these cases the kernel is the subgroup $\{1, 4\}$.

   (b) Give an example of a group G and a subgroup N such that N is *not* normal in G. Explain your answer.
   The subgroup $H = \langle b \rangle$ is not normal in S_3 since its left and right cosets are different: $aH = \{a \cdot e, a \cdot b\} = \{a, b\}$ but $Ha = \{e \cdot a, b \cdot a\} = \{a, a^2 b\}$.

   (c) Show that $GL_2(\mathbf{Z}_2)$ is isomorphic to S_3.
   A 2×2 matrix is invertible iff its rows are linearly independent. There are 3 nonzero rows over $\mathbf{Z}_2$, and then the second row must be nonzero and different, giving a total of 6 matrices in $GL_2(\mathbf{Z}_2)$. Since $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ but $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, the group is not abelian. By your homework problems, any nonabelian group of order 6 has to be isomorphic to S_3.

   (d) Show that $GL_n(\mathbf{R})/SL_n(\mathbf{R}) \cong \mathbf{R}^\times$. See Example 3.8.14.

   (e) Show that $(Z \times Z) \,/\, \langle (1,1) \rangle$ is an infinite cyclic group.
   No nonzero multiple of the element $(1, 0)$ is in $\langle (1,1) \rangle$, so the coset $(1,0) + \langle (1,1) \rangle$ has infinite order. We also have $(m, n) + \langle (1,1) \rangle = (m - n, 0) + \langle (1,1) \rangle = m((1,0) + \langle (1,1) \rangle)$, which shows that $(1,0) + \langle (1,1) \rangle$ is a generator for the group. Conclusion: $(Z \times Z) \,/\, \langle (1,1) \rangle$ is cyclic and infinite.