"Linear Algebra" is the study of linear functions and operations. In Calculus, functions of the form $f(x) = mx + b$ are called linear. In this course, we focus on similar functions, with more variables, but with a minor change in terminology. The important part of a linear function is the multiplication, and in our new terminology, only functions of the form $f(x) = mx$ will be called linear. We restrict our definition to functions whose graph is a straight line *through the origin*. To show that the input is a single variable, and the output is a single variable, we write $f : \mathbf{R} \to \mathbf{R}$.

If we allow two input variables, then we have a function from $\mathbf{R}^2$ to $\mathbf{R}$. If the function is linear, then its graph should be a plane through the origin, with $f(x_1, x_2) = m_1x_1 + m_2x_2$. What would a linear function look like if we allow both the input and the output to have two variables? We use the notation $f : \mathbf{R}^2 \to \mathbf{R}^2$, with the formula defined by $f(x_1, x_2) = (m_{11}x_1 + m_{12}x_2,\, m_{21}x_1 + m_{22}x_2)$. If we change the row vectors to column vectors, and use matrix notation, we can rewrite the function in the form

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} m_{11}x_1 + m_{12}x_2 \\ m_{21}x_1 + m_{22}x_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

Finally, to make this look more like the one-dimensional example, we can use $\mathbf{x}$ for the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and M for the matrix $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$, so that the notation becomes simply $f(\mathbf{x}) = M\mathbf{x}$.

Some important examples:

(1) Transportation problems: If a company has five manufacturing plants and twelve warehouses spread across the U.S., the problem it faces is to meet the demand at each of the warehouses, while minimizing transportation costs from the plants to the warehouses. The variable cost in the shipping is roughly proportional to the distances involved, and so the variable part of the cost function is a linear function. Matrix techniques have been developed that can be used to solve this minimization problem.

(2) Linear approximations: In Calculus, we use tangent lines to give linear approximations to a function. That is, if $f : \mathbf{R} \to \mathbf{R}$, and $x \sim a$, then $f(x) \sim f'(a)(x - a) + f(a)$. To find a local linear approximation to a function $f : \mathbf{R}^3 \to \mathbf{R}^3$, for example, we replace the variable x with a 3-dimensional vector, and then the "linear" part $f'(a)(x - a)$ is replaced by a matrix product in which $f'(a)$ becomes a 3×3 matrix whose 9 entries are the possible partial derivatives.

Some additional applications:

(3) analytic geometry (putting conic sections and quadric surfaces in standard form);

(4) computer graphics (rotating a figure in 3 dimensions can be done by multiplying its coordinates by an appropriate matrix);

(5) probability theory (Markov chains);

(6) economics (Leontief input–output models);

(7) least squares fitting of data;

(8) CAT scans (X-ray pictures taken from various angles must be combined to reconstruct a two-dimensional picture);

(9) data transmission (several important encryption techniques use matrices).

We will also study general linear processes. A linear function of the form $f(x) = mx$ has two important properties that force its graph to be a straight line:

$$f(x_1 + x_2) = m(x_1 + x_2) = f(x_1) + f(x_2) \qquad \text{and} \qquad f(kx) = m(kx) = k(mx) = kf(x) .$$

The processes of differentiation and integration are also "linear" operations because they respect addition and multiplication by scalars: $\dfrac{d}{dx}(f_1(x) + f_2(x)) = \dfrac{d}{dx}f_1(x) + \dfrac{d}{dx}f_2(x)$ and $\dfrac{d}{dx}(kf(x)) = k\,\dfrac{d}{dx}f(x)$; and also $\displaystyle\int_a^b (f_1(x) + f_2(x))\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx$ and $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$.

We will study "vector spaces" so that we have a context in which to study "linear transformations", which respect addition of vectors and scalar multiplication of vectors, and therefore satisfy these equations:

$$T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w}) \qquad \text{and} \qquad T(k\mathbf{v}) = kT(\mathbf{v}) .$$