MATH 423 TEST II 7/17/2006

Professor John Beachy

Some notation: $M_n(F)$ denotes the vector space of $n \times n$ matrices over the field F. For a matrix A: ${}^tA$ denotes the transpose, $D(A)$ the determinant, and A^* the adjoint.

1. (20 pts) (a) Define the *rank* and *nullity* of a linear transformation.

(b) Define what it means for two matrices to be *similar*.

(c) State Cramer's rule for the solution of a system of linear equations.

(d) Complete the following definition: Let F be an arbitrary field. A *determinant* is a function which assigns to each n-tuple $\{a_1, \ldots, a_n\}$ of vectors in F_n an element of F, denoted by $D(a_1, \ldots, a_n)$, such that the following conditions are satisfied:

2. (20 pts) (a) Let A be a fixed $n \times n$ matrix, and define $T : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ by $T(X) = AX - XA$, for all $X \in M_n(\mathbf{R})$. Check that T defines a linear transformation.

(b) Given the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, choose a basis for $M_2(\mathbf{R})$ and find the matrix of the linear transformation T defined in part (a), relative to the basis you have chosen.

(c) Find the rank of T, and the rank of T^2.

(d) Find the null space $n(T)$ and the range $T(M_2(\mathbf{R}))$ of T (in each case expressed as a set of 2×2 matrices).

3. (20 pts) State and prove the rank-nullity theorem for a linear transformation $T : V \to W$. *Hint:* This relates the rank and nullity of T to the dimension of V.

4. (10 pts) Let A be an $n \times n$ matrix. Show that if A^{-1} exists and ${}^tA = A^{-1}$, then $D(A) = \pm 1$.

5. (20 pts) Let A be an $n \times n$ matrix.

(a) Show that $({}^tA)^* = {}^t(A^*)$.

(b) Show that if A is symmetric, then so is A^*.

(c) Find the adjoint of the skew symmetric matrix $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

(d) Find (with proof) a formula for $(-A)^*$.

(e) Conjecture (and prove) the result analogous to (b) for skew symmetric matrices.

6. (10 pts) Let V be a finite dimensional vector space over the field F, and let $T : V \to V$ be a linear transformation. Prove that if T^2 has the same rank as T, then $V = n(T) \oplus T(V)$.