1. (20 pts) (a) Define the rank and nullity of a linear transformation.

(b) Define what it means for two matrices to be similar.

(c) State Cramer’s rule for the solution of a system of linear equations.

(d) Complete the following definition: Let $F$ be an arbitrary field. A determinant is a function which assigns to each $n$-tuple $\{a_1, \ldots, a_n\}$ of vectors in $F$ an element of $F$, denoted by $D(a_1, \ldots, a_n)$, such that the following conditions are satisfied:

2. (20 pts) (a) Let $A$ be a fixed $n \times n$ matrix, and define $T : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ by $T(X) = AX -XA$, for all $X \in M_n(\mathbb{R})$. Check that $T$ defines a linear transformation.

(b) Given the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, choose a basis for $M_2(\mathbb{R})$ and find the matrix of the linear transformation $T$ defined in part (a), relative to the basis you have chosen.

(c) Find the rank of $T$, and the rank of $T^2$.

(d) Find the null space $n(T)$ and the range $T(M_2(\mathbb{R}))$ of $T$ (in each case expressed as a set of $2 \times 2$ matrices).

3. (20 pts) State and prove the rank-nullity theorem for a linear transformation $T : V \rightarrow W$. Hint: This relates the rank and nullity of $T$ to the dimension of $V$.

4. (10 pts) Let $A$ be an $n \times n$ matrix. Show that if $A^{-1}$ exists and $^tA = A^{-1}$, then $D(A) = \pm 1$.

5. (20 pts) Let $A$ be an $n \times n$ matrix.

(a) Show that $(^tA)^* = (^t(A^*))$.

(b) Show that if $A$ is symmetric, then so is $A^*$.

(c) Find the adjoint of the skew symmetric matrix $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

(d) Find (with proof) a formula for $(-A)^*$.

(e) Conjecture (and prove) the result analogous to (b) for skew symmetric matrices.

6. (10 pts) Let $V$ be a finite dimensional vector space over the field $F$, and let $T : V \rightarrow V$ be a linear transformation. Prove that if $T^2$ has the same rank as $T$, then $V = n(T) \oplus T(V)$. 