MATH 423 TEST III 7/31/2006

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The underlying assumption is that V is a finite dimensional vector space.

1. (15 pts) Prove **either** *Part* A **OR** *Part B.*

Part A: (Lemma 15.8) Every orthonormal set of vectors is a linearly independent set.

Part B: (Theorem 22.8) Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be eigenvectors belonging to distinct eigenvalues $\alpha_1, \ldots, \alpha_r$ of $T \in L(V, V)$. Then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent.

2. (15 pts) Let $T \in L(V, V)$, and let $m(x)$ be the minimal polynomial of T. Prove the proposition from the class notes which states that $m(x)$ has the same zeros as the characteristic polynomial of T.

3. (10 pts) Find the characteristic polynomial and minimal polynomial (over $\mathbf{R}$) of each of the following matrices:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

4. (15 pts) over the field $\mathbf{R}$ of real numbers, find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$.

5. (15 pts) In $\mathbf{R}_3$, let $\mathbf{u}_1 = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$, $\mathbf{u}_2 = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$, $\mathbf{w}_3 = (1, 1, 1)$.

(a) Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_3\}$ into an orthonormal basis.

(b) Find the coordinates of $(1, 0, 1)$ relative to the orthonormal basis in part (a).

6. (10 pts) Let V be an inner product space, and let $T \in L(V, V)$ be an orthogonal transformation. Prove that if W is a T-invariant subspace of V, then $W^\perp$ is also a T-invariant subspace of V. (Recall that $W^\perp = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}$.)

7. (20 pts) Let $T \in L(V, V)$.

(a) Assume that $V = W_1 \oplus W_2$, where W_1 and W_2 are T-invariant subspaces of V. Show that it is possible to find a basis for V such that the matrix of T relative to this basis has "block diagonal form" $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$.

(b) If T^2 has the same rank as T, apply part (a) to find a matrix for T in simplified form.