Choose 3 of the first 4 problems, for 20 points each.
Work problem 5, for 40 points (the parts can be done independently).

The underlying assumption is that $V$ is a finite dimensional vector space.

Choose 3 of the following 4 problems (60 pts). I recommend 1–3.

1. Let $T \in L(V, V)$.
   
   (a) Define: $T$-invariant subspace; generalized eigenspace of $T$.
   
   (b) Prove that each generalized eigenspace of $T$ is a $T$-invariant subspace of $V$.

2. Let $A$ be a $10 \times 10$ matrix with real entries, whose characteristic polynomial is $(x - 3)^{10}$. Suppose that the nullspace $n(A - 3I)$ has dimension 5, that $n((A - 3I)^4)$ has dimension 10, and that $n((A - 3I)^3)$ has dimension less than 10.

   (a) In the Jordan canonical form for $A$, how many secondary blocks will there be? Explain what a secondary block is, and explain how you got your answer.

   (b) List all of the possible Jordan forms for $A$. Give a brief explanation for your answer.

3. Let $D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ be the matrix that represents differentiation (from polynomials of degree $\leq 3$ to polynomials of degree $\leq 3$). Find the matrix which gives the Jordan canonical form for $D$. Give an explanation for your answer.

4. Let $W_1 \subseteq W_2$ be subspaces of $V$. Give the definition of a complementary basis for $W_1$ in $W_2$. Then prove the following lemma from the class notes:

   Let $L : V \to V$ be a linear transformation. Let $\{u_1, \ldots, u_k\}$ be a complementary basis for $n(L^n)$ in $n(L^{n+1})$. Then $\{L(u_1), \ldots, L(u_k)\}$ is part of a complementary basis for $n(L^{n-1})$ in $n(L^n)$.

5. (40 pts) Let $A = \begin{bmatrix} 0 & -8 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ -1 & -3 & 2 & 1 \\ -2 & -8 & 1 & 4 \end{bmatrix}$

   (a) Find the characteristic polynomial of $A$ by calculating $D(xI - A)$. (Answer: $(x - 2)^4$)

   (b) Find the dimension of $n(A - 2I)$. (Answer: 2)

   (c) Find the dimension of $n((A - 2I)^2)$. (Answer: 3)

   (d) Give the Jordan canonical form for $A$, and explain your answer.