Chapter Summaries for Kolman / Hill, *Elementary Linear Algebra*, 8th Ed.
Sections 1.1–1.6, 2.1–2.2, 3.2–3.8, 4.3–4.5, 5.1–5.3, 5.5, 6.1–6.5, 7.1–7.2, 7.4

Chapter 1: Linear Equations and Matrices

DEFINITIONS

There are a number of definitions concerning systems of linear equations on pages 1–2 and 26–27. The numbered definitions **1.1**–**1.4** and **1.6**–**1.7** concern matrices and matrix operations.

1.5 (p 18). If $A = [a_{ij}]$ is an $m \times n$ matrix, then its **transpose** is $A^T = [a_{ji}]$.

1.8–**1.9** (p 40). A square matrix A is **symmetric** if $A^T = A$ and **skew symmetric** if $A^T = -A$.

1.10 (p 43). An $n \times n$ matrix A is **invertible** (also called **nonsingular**) if it has an **inverse** matrix A^{-1} such that $AA^{-1} = I_n$ and $A^{-1}A = I_n$.
If A is not invertible, it is called a **singular** matrix.

(p 30 Ex #41). The **trace** of a square matrix is the sum of the entries on its main diagonal.

THEOREMS

1.1–**1.4** (pp 32–35) These theorems list the basic properties of matrix operations. The only surprises are that we can have $AB \neq BA$, and that $(AB)^T = B^T A^T$.

1.5–**1.8** (pp 43–45) Inverses are unique. If A and B are invertible $n \times n$ matrices, then AB, A^{-1}, and A^T are invertible, with $(AB)^{-1} = B^{-1}A^{-1}$, $(A^{-1})^{-1} = A$, and $(A^T)^{-1} = (A^{-1})^T$.

USEFUL EXERCISES

(p 48 Ex #42). The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, and in this case $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Chapter 2: Solving Linear Systems

DEFINITIONS

2.1 (p 78). The definitions of **row echelon form** and **reduced row echelon form** are easier to understand than to write down. You need to know how to use them.

2.2 (p 79) These are the **elementary row operations** on a matrix: (I) interchange two rows; (II) multiply a row by a nonzero number; (III) add a multiple of one row to another.

2.3 (p 80) Two matrices are **row equivalent** if one can be obtained from the other by doing a finite sequence of elementary row operations.

2.4 (p 101) A matrix that is obtained from the identity matrix by doing a single elementary row operation is called an **elementary matrix**.

THEOREMS

2.1–2.2 (pp 80–83). Every matrix is row equivalent to a matrix in row echelon form, and row equivalent to a *unique* matrix in reduced row echelon form.

2.3 (p 85). Linear systems are equivalent if and only if their augmented matrices are row equivalent.

2.4 (p 95). A homogeneous linear system has a nontrivial solution whenever it has more unknowns than equations.

2.5–2.6 (p 102). Elementary row operations can be done by multiplying by elementary matrices.

2.7 (p 102). Any elementary matrix has an inverse that is an elementary matrix of the same type.

2.8 (p 103). A matrix is invertible if and only if it can be written as a product of elementary matrices.

2.9 (p 102). The homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if A is singular.

2.10 (p 106). A square matrix is singular if and only if it is row equivalent to a matrix with a row of zeros.

2.11 (p 108). If A and B are $n \times n$ matrices with $AB = I_n$, then $BA = I_n$ and thus $B = A^{-1}$.

The summary on page 104 is important:

The following conditions are equivalent for an $n \times n$ matrix A:
(1) A is invertible;
(2) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution;
(3) A is row equivalent to the identity;
(4) for every vector $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution;
(5) A is a product of elementary matrices.

ALGORITHMS

1. (pp 85–91) *Gauss-Jordan reduction*. This algorithm puts a matrix in reduced row echelon form. You need to know how to it quickly and accurately. *Gaussian elimination* uses back substitution after putting the matrix in row echelon form.

2. (pp 105–106) To find the inverse A^{-1} of an invertible matrix A, start with $[A|I_n]$ and use Gauss-Jordan reduction to reduce it to $[I_n|A^{-1}]$.

2. (p 103) To write an invertible matrix A as a product of elementary matrices, row reduce it to the identity and keep track of the corresponding elementary matrices. If $E_k \cdots E_2 E_1 A = I_n$, then $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$.

Chapter 3: Real Vector Spaces

DEFINITIONS

3.4 (p 135). A **vector space** is a set V on which two operations are defined, called **vector addition** and **scalar multiplication**, and denoted by $+$ and $\cdot$, respectively.
The operation $+$ (vector addition) must satisfy the following conditions:

Closure: For all vectors $\mathbf{u}$ and $\mathbf{v}$ in V, the sum $\mathbf{u}+\mathbf{v}$ belongs to V.

(1) *Commutative law:* For all vectors $\mathbf{u}$ and $\mathbf{v}$ in V, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

(2) *Associative law:* For all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V, $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$.

(3) *Additive identity:* The set V contains an **additive identity** element, denoted by $\mathbf{0}$, such that for all vectors $\mathbf{v}$ in V, $\mathbf{0}+\mathbf{v}=\mathbf{v}$ and $\mathbf{v}+\mathbf{0}=\mathbf{v}$.

(4) *Additive inverses:* For each vector $\mathbf{v}$ in V, the equations $\mathbf{v}+\mathbf{x}=\mathbf{0}$ and $\mathbf{x}+\mathbf{v}=\mathbf{0}$ have a solution $\mathbf{x}$ in V, called an **additive inverse** of $\mathbf{v}$, and denoted by $-\mathbf{v}$.

The operation $\cdot$ (scalar multiplication) must satisfy the following conditions:

Closure: For all real numbers c and all vectors $\mathbf{v}$ in V, the product $c\cdot\mathbf{v}$ belongs to V.

(5) *Distributive law:* For all real numbers c and all vectors $\mathbf{u}$, $\mathbf{v}$ in V, $c\cdot(\mathbf{u}+\mathbf{v})=c\cdot\mathbf{u}+c\cdot\mathbf{v}$.

(6) *Distributive law*: For all real numbers c, d and all vectors $\mathbf{v}$ in V, $(c+d)\cdot\mathbf{v}=c\cdot\mathbf{v}+d\cdot\mathbf{v}$.

(7) *Associative law*: For all real numbers c, d and all vectors $\mathbf{v}$ in V, $c\cdot(d\cdot\mathbf{v})=(cd)\cdot\mathbf{v}$.

(8) *Unitary law*: For all vectors $\mathbf{v}$ in V, $1\cdot\mathbf{v}=\mathbf{v}$.

3.5 (p 142). Let V be a vector space, and let W be a subset of V. If W is a vector space with respect to the operations in V, then W is called a **subspace** of V.

3.6 (p 145). A vector $\mathbf{v}$ is a **linear combination** of the vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k$ if there are real numbers $a_1,a_2,\ldots,a_k$ with $\mathbf{v}=a_1\mathbf{v}_1+a_2\mathbf{v}_2+\ldots+a_k\mathbf{v}_k$.

3.7 (p 146). The set of all linear combinations of the vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k$ is the **span** of the vectors, denoted by $\text{span}\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$.

Def (p 148). For an $m\times n$ matrix A, the **null space** of A is $\{\mathbf{x}\mid A\mathbf{x}=\mathbf{0}\}$.

3.8 (p 154). Vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k$ **span** the vector space V if $\text{span}\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}=V$.

3.9 (p 156). The vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k$ are **linearly independent** if the equation
$x_1\mathbf{v}_1+x_2\mathbf{v}_2+\ldots+x_k\mathbf{v}_k=\mathbf{0}$ has only the trivial solution (all zeros).
The vectors are **linearly dependent** if the equation has a nontrivial solution (not all zeros).

3.10 (p 164). A set of vectors in V is a **basis** for V if it is a linearly independent spanning set.

3.11 (p 172). A vector space V has **dimension** n, denoted $\dim(V)=n$, if it has a basis with n elements.

Def (p 186). Let $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ be a basis for V. If $\mathbf{v}=a_1\mathbf{v}_1+a_2\mathbf{v}_2+\ldots+a_n\mathbf{v}_n$, then the *column* vector $[\mathbf{v}]_S$ with entries $a_1,a_2,\ldots,a_n$ is called the **coordinate vector of v with respect to the ordered basis** S**.**

Def (p 193). If S,T are bases for V, then the **transition matrix** $P_{S\leftarrow T}$ from T to S is defined by the equation $[\mathbf{v}]_S=P_{S\leftarrow T}\cdot[\mathbf{v}]_T$.

3.14 (p 201). The subspace spanned by the rows of a matrix is called its **row space**; the subspace spanned by the columns is its **column space**.

3.15 (p 204). The **row rank** of a matrix is the dimension of its row space; the **column rank** is the dimension of its column space.

THEOREMS

3.1–3.2 (pp 133,140). These theorems deal with properties of vectors.

3.3 (p 142). Let V be a vector space, with operations $+$ and $\cdot$, and let W be a subset of V. Then W is a subspace of V if and only if the following conditions hold.

W is nonempty: The zero vector belongs to W.

Closure under $+$: If $\mathbf{u}$ and $\mathbf{v}$ are any vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.

Closure under $\cdot$: If $\mathbf{v}$ is any vector in W, and c is any real number, then $c\cdot\mathbf{v}$ is in W.

3.4 (p 147). For any vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k$, the subset $\text{span}\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ is a subspace.

3.6 (p 160). A set of vectors is linearly dependent if and only if one is a linear combination of the rest.
Note: This theorem is probably the best way to think about linear dependence, although the usual definition does give the best way to check whether or not a set of vectors is linearly independent.

3.7 (p 167). A set of vectors is a basis for the vector space V if and only if every vector in V can be expressed uniquely as a linear combination of the vectors in the set.

3.8 (p 168). Any spanning set contains a basis.

3.9 (p 171). If a vector space has a basis with n elements, then it cannot contain more than n linearly independent vectors.

Cor (pp 172–173). Any two bases have the same number of elements;
$\dim(V)$ is the maximum number of linearly independent vectors in V;
$\dim(V)$ is the minimum number of vectors in any spanning set.

3.10 (p 174). Any linearly independent set can be expanded to a basis.

3.11 (p 175). If $\dim(V) = n$, and you have a set of n vectors, then to check that it forms a basis you only need to check *one* of the two conditions (spanning and linear independence).

3.16 (p 202). Row equivalent matrices have the same row space.

3.17 (p 206). For any matrix (of any size), the row rank and column rank are equal.

3.18 [The rank-nullity theorem] (p 207). If A is any $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = n$.

3.20 (p 209). An $n \times n$ matrix has rank n if and only if it is row equivalent to the identity.

3.20 (p 210). $A\mathbf{x} = \mathbf{b}$ has a solution if and only if the augmented matrix has the same rank as A.

Summary of results on invertibility (p 211). The following are equivalent for any $n \times n$ matrix A:
(1) A is invertible (3) A is row equivalent to the identity (5) A is a product of elementary matrices
(4) $A\mathbf{x} = \mathbf{b}$ always has a solution (2) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
(6) $\text{rank}(A) = n$ (7) $\text{nullity}(A) = 0$
(8) the rows of A are linearly independent (9) the columns of A are linearly independent

ALGORITHMS

1. (p 156). To test whether the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent or linearly dependent:
Solve the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_k\mathbf{v}_k = \mathbf{0}$. *Note: the vectors must be* columns *in a matrix.*
If the only solution is all zeros, then the vectors are *linearly independent.*
If there is a nonzero solution, then the vectors are *linearly dependent.*

2. (p 154). To check that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ span the subspace W:
Show that for every vector $\mathbf{b}$ in W there is a solution to $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_k\mathbf{v}_k = \mathbf{b}$.

3. (p 170). To find a basis for the subspace $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ by deleting vectors:
(i) Construct the matrix whose *columns* are the coordinate vectors for the $\mathbf{v}$'s
(ii) Row reduce
(iii) Keep the vectors whose column contains a leading 1
Note: The advantage here is that the answer consists of some of the vectors in the original set.

4. (p 174). To find a basis for a vector space that includes a given set of vectors, expand the set to include all of the standard basis vectors and use the previous algorithm.

5. (p 178). To find a basis for the solution space of the system $A\mathbf{x} = \mathbf{0}$:
(i) Row reduce A
(ii) Identify the independent variables in the solution
(iii) In turn, let one of these variables be 1, and all others be 0
(iv) The corresponding solution vectors form a basis

6. (p 184). To solve $A\mathbf{x} = \mathbf{b}$, find one particular solution $\mathbf{v}_p$ and add to it all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

7. (p 193). To find the transition matrix $P_{S \leftarrow T}$:
Note: The purpose of the procedure is to allow a change of coordinates $[\mathbf{v}]_S = P_{S \leftarrow T} \cdot [\mathbf{v}]_T$.
(i) Construct the matrix A whose *columns* are the coordinate vectors for the basis S
(ii) Construct the matrix B whose *columns* are the coordinate vectors for the basis T
(iii) Row reduce the matrix $[\ A \mid B\ ]$ to get $[\ I \mid P_{S \leftarrow T}\ ]$
Shorthand notation: Row reduce $[\ S \mid T\ ] \rightsquigarrow [\ I \mid P_{S \leftarrow T}\ ]$

8. (p 202) To find a simplified basis for the subspace $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$:
(i) Construct the matrix whose *rows* are the coordinate vectors for the $\mathbf{v}$'s
(ii) Row reduce
(iii) The nonzero rows form a basis
Note: The advantage here is that the vectors have lots of zeros, so they are in a useful form.

Chapter 4: Inner Product Spaces

DEFINITIONS

4.1 (p 235). Let V be a real vector space. An **inner product** on V is a function that assigns to each ordered pair of vectors $\mathbf{u}, \mathbf{v}$ in V a real number $(\mathbf{u}, \mathbf{v})$ satisfying:

(1) $(\mathbf{u}, \mathbf{u}) \geq 0$; $(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
(2) $(\mathbf{v}, \mathbf{u}) = (\mathbf{u}, \mathbf{v})$ for any $\mathbf{u}, \mathbf{v}$ in V.
(3) $(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$ for any $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V.
(4) $(c\mathbf{u}, \mathbf{v}) = c(\mathbf{u}, \mathbf{v})$ for any $\mathbf{u}, \mathbf{v}$ in V and any scalar c.

4.2 (p 240). A vector space with an inner product is called an **inner product space**. It is a **Euclidean space** if it is also finite dimensional.

Def (p 239). If V is a Euclidean space with ordered basis S, then the matrix C with $(\mathbf{v}, \mathbf{w}) = [\mathbf{v}]_S^T \, C \, [\mathbf{w}]_S$ is called the matrix of the inner product with respect to S (see Theorem 4.2).

4.3 (p 243). The distance between vectors $\mathbf{u}$ and $\mathbf{v}$ is $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$.

4.4 (p 243). The vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $(\mathbf{u}, \mathbf{v}) = 0$.

4.5 (p 243). A set S of vectors is **orthogonal** if any two distinct vectors in S are orthogonal. If each vector in S also has length 1, then S is **orthonormal**.

4.6 (p 260). The **orthogonal complement** of a subset W is $W^\perp = \{\mathbf{v} \text{ in } V \mid (\mathbf{v}, \mathbf{w}) = 0 \text{ for all } \mathbf{w} \text{ in } V\}$.

Def (p 267). Let W be a subspace with orthonormal basis $\mathbf{w}_1, \dots, \mathbf{w}_m$. The **orthogonal projection** of a vector $\mathbf{v}$ on W is $\text{proj}_W(\mathbf{v}) = \sum_{i=1}^m (\mathbf{v}, \mathbf{w}_i)\mathbf{w}_i$.

THEOREMS

4.2 (p 237). Let V be a Euclidean space with ordered basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$. If C is the matrix $[(\mathbf{u}_i, \mathbf{u}_j)]$, then for every $\mathbf{v}, \mathbf{w}$ in V the inner product is given by $(\mathbf{v}, \mathbf{w}) = [\mathbf{v}]_S^T \, C \, [\mathbf{w}]_S$.

4.3 (p 240). If $\mathbf{u}, \mathbf{v}$ belong to an inner product space, then $|(\mathbf{u}, \mathbf{v}) \leq ||\mathbf{u}|| \, ||\mathbf{v}||$. (Cauchy–Schwarz inequality)

Cor (p 242). If $\mathbf{u}, \mathbf{v}$ belong to an inner product space, then $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$. (Triangle inequality)

4.4 (p 244). A finite orthogonal set of nonzero vectors is linearly independent.

4.5 (p 248). Relative to an orthonormal basis, coordinate vectors can be found by using the inner product.

4.6 (p 249). Any finite dimensional subspace of an inner product space has an orthonormal basis.

4.7 (p 253). If S is an orthonormal basis for an inner product space V, then $(\mathbf{u}, \mathbf{v}) = [\mathbf{u}]_S \cdot [\mathbf{v}]_S$.

4.9 (p 260). If W is a subspace of V, then $W^\perp$ is a subspace with $W \cap W^\perp = \{\mathbf{0}\}$.

4.10 (p 262). If W is a finite dimensional subspace of an inner product space V, then every vector in V can be written uniquely as a sum of a vector in W and a vector in $W^\perp$. Notation: $V = W \oplus W^\perp$

4.12 (p 263). The null space of a matrix is the orthogonal complement of its row space.

ALGORITHMS

1. (p 249). To find an orthonormal basis for a finite dimensional subspace W: (Gram–Schmidt process)

(i) Start with any basis $S = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ for W

(ii) Start constructing an orthogonal basis $T^* = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ for W by letting $\mathbf{v}_1 = \mathbf{u}_1$.

(iii) To find $\mathbf{v}_2$, start with $\mathbf{u}_2$ and subtract its projection onto $\mathbf{v}_1$.

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{(\mathbf{u}_2, \mathbf{v}_1)}{(\mathbf{v}_1, \mathbf{v}_1)}\mathbf{v}_1$$

(iv) To find $\mathbf{v}_i$, start with $\mathbf{u}_i$ and subtract its projection onto the span of the previous $\mathbf{v}_i$'s

$$\mathbf{v}_i = \mathbf{u}_i - \frac{(\mathbf{u}_i, \mathbf{v}_1)}{(\mathbf{v}_1, \mathbf{v}_1)}\mathbf{v}_1 - \frac{(\mathbf{u}_i, \mathbf{v}_2)}{(\mathbf{v}_2, \mathbf{v}_2)}\mathbf{v}_2 - \dots - \frac{(\mathbf{u}_i, \mathbf{v}_{i-1})}{(\mathbf{v}_{i-1}, \mathbf{v}_{i-1})}\mathbf{v}_{i-1}$$

(v) Construct the orthonormal basis $T = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ by dividing each $\mathbf{v}_i$ by its length

$$\mathbf{w}_i = \frac{1}{||\mathbf{v}_i||}\mathbf{v}_i$$

Chapter 5: Linear Transformations and Matrices

DEFINITIONS

5.1 (p 288). A function $L : V \to W$ between vector spaces is a **linear transformation** if
$L(c\mathbf{u} + d\mathbf{v}) = cL(\mathbf{u}) + dL(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V and all real numbers c, d.

5.2 (p 299). A linear transformation $L : V \to W$ is **one-to-one** if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies $\mathbf{v}_1 = \mathbf{v}_2$.

5.3 (p 300). The **kernel** of a linear transformation $L : V \to W$ is $\ker(L) = \{\mathbf{v} \text{ in } V \mid L(\mathbf{v}) = \mathbf{0}\}$.

5.4 (p 302). The **range** of a linear transformation $L : V \to W$ is $\text{range}(L) = \{L(\mathbf{v}) \mid \mathbf{v} \text{ is in } V\}$.
The linear transformation L is called **onto** if $\text{range}(L) = W$.

Def (p 318). Let V be a vector space with basis S, let W be a vector space with basis T, and let $L : V \to W$ be a linear transformation. Then the matrix A **represents** L (with respect to S and T) if
$[L(\mathbf{x})]_T = A \cdot [\mathbf{x}]_S$ for all $\mathbf{x}$ in V. Notation: $A = M_{T \leftarrow S}(L)$, and so $[L(\mathbf{x})]_T = M_{T \leftarrow S}(L) \cdot [\mathbf{x}]_S$.

(p 294) If $L : \mathbf{R}^n \to \mathbf{R}^m$, and S and T are the standard bases for $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively, then A is called the **standard matrix** representing L.

5.6 (p 335). The square matrices A, B are **similar** if there is an invertible matrix P that is a solution of the equation $B = X^{-1}AX$.

THEOREMS

5.2 (p 293). A linear transformation is completely determined by what it does to basis vectors.

5.4 (p 301). The kernel of a linear transformation $L : V \to W$ is a subspace of V, and L is one-to-one if and only if its kernel is zero.

5.5 (p 303). The range of a linear transformation $L : V \to W$ is a subspace of W.

5.6 (p 306). If $L : V \to W$ is a linear transformation, and V has finite dimension, then
$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(V)$. Note: This is a version of the rank-nullity theorem (3.18).

5.7 (p 309) A linear transformation is invertible if and only if it is one-to-one and onto.

(p 309) If $L : V \to W$ and $\dim V = \dim W$, then L is one-to-one if and only if it is onto.

5.9 (p 314). If $L : V \to W$ is a linear transformation, and V and W have finite bases S and T, then the columns of the matrix $M_{T \leftarrow S}(L)$ that represents L with respect to S and T are found by applying L to the basis vectors in S.
Shorthand notation: $M_{T \leftarrow S}(L) = [\ L(S)\ ]_T$

5.12 (p 331). Let $L : V \to W$ be a linear transformation, and suppose that S, S' are bases for V and T, T' are bases for W. Then the matrix $M_{T' \leftarrow S'}(L)$ representing L with respect to S' and T' is related to the matrix $M_{T \leftarrow S}(L)$ representing L with respect to S and T by the equation

$$M_{T' \leftarrow S'}(L) = P_{T' \leftarrow T} \cdot M_{T \leftarrow S}(L) \cdot P_{S \leftarrow S'}$$

where $P_{S \leftarrow S'}$ is the transition matrix that changes coordinates relative to S' into coordinates relative to S and $P_{T' \leftarrow T}$ is the transition matrix that changes coordinates relative to T into coordinates relative to T'.

Cor (p 334). Let $L : V \to V$ be a linear transformation, and suppose that S, S' are bases for V. Then
$B = P^{-1}AP$ for $B = M_{S' \leftarrow S'}(L)$, $A = M_{S \leftarrow S}(L)$, and $P = P_{S \leftarrow S'}$.

5.13 (p 334). If $L : V \to V$ is a linear transformation represented by A, then $\dim(\text{range}(L)) = \text{rank}(A)$.

5.14 (p 336). Square matrices are similar if and only if they represent the same linear transformation.

5.15 (p 337). Similar matrices have the same rank.

ALGORITHMS

1. (p 316). To find the matrix $M_{T \leftarrow S}(L)$ that represents L with respect to the bases S and T:
(i) Find $L(\mathbf{v})$ for each basis vector $\mathbf{v}$ in S.
(ii) For each $\mathbf{v}$ in S, find the coordinate vectors $[L(\mathbf{v})]_T$ of $L(\mathbf{v})$ with respect to the basis T.
(iii) Put in the coordinate vectors $[L(\mathbf{v})]_T$ as *columns* of $M_{T \leftarrow S}(L)$.
Shorthand notation: Row reduce $[\ T \mid L(S)\ ] \rightsquigarrow [\ I \mid M_{T \leftarrow S}(L)\ ]$

2. (p 305). To find a basis for the kernel of a linear transformation $L : V \to W$:
(i) Choose bases for V and W and find the matrix A that represents L.
(ii) Find a basis for the *solution space* of the system $A\mathbf{x} = \mathbf{0}$.

3. (p 305). To find a basis for the range of a linear transformation $L : V \to W$:
(i) Choose bases for V and W and find a matrix A that represents L.
(ii) Find a basis for the *column* space of A.

Chapter 6: Determinants

Important note: the definitions and theorems in this chapter apply to *square* matrices.

DEFINITIONS

6.4 (p 373). Let $A = [a_{ij}]$ be a matrix. The **cofactor** of a_{ij} is $A_{ij} = (-1)^{i+j}\det(M_{ij})$, where M_{ij} is the matrix found by deleting the ith row and jth column of A.

6.5 (p 381). The **adjoint** of a matrix A is $\mathrm{adj}(A) = [A_{ji}]$, where A_{ij} is the cofactor of a_{ij}.

THEOREMS

Results connected to row reduction:

6.2 (p 363). Interchanging two rows (or columns) of a matrix changes the sign of its determinant.

6.5 (p 365). If every entry in a row (or column) has a factor k, then k can be factored out of the determinant.

6.6 (p 365). Adding a multiple of one row (or column) to another does not change the determinant.

6.7 (p 366). If A is in row echelon form, then $\det(A)$ is the product of terms on the main diagonal.

Some consequences:

6.3 (p 364). If two rows (or columns) of a matrix are equal, then its determinant is zero.

6.4 (p 364). If a matrix has a row (or column) of zeros, then its determinant is zero.

Other facts about the determinant:

6.1 (p 363). For the matrix A, we have $\det(A^T) = \det(A)$.

6.8 (p 369). The matrix A is invertible if and only if $\det(A) \neq 0$.

6.9 (p 369). If A and B are both $n \times n$ matrices, then $\det(AB) = \det(A) \cdot \det(B)$.

Cor (p 370). If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Cor (p 370). Similar matrices have the same determinant.

Expansion by cofactors:

6.10 (p 374). If A is an $n \times n$ matrix, then $\det(A) = \sum_{j=1}^{n} a_{ij}A_{ij}$. (expansion by cofactors along row i)
Note: This can be used to define the determinant by induction, rather than as in Def 6.2 on p 359.

6.12 (p 382). For the matrix A we have $A \cdot \mathrm{adj}(A) = \det(A) \cdot I$ and $\mathrm{adj}(A) \cdot A = \det(A) \cdot I$.

Cor (p 383). If A is invertible, then $A^{-1} = \dfrac{1}{\det(A)}\,\mathrm{adj}(A)$.

5.13 [Cramer's rule] (p 385). If $\det(A) \neq 0$, then the system $A\mathbf{x} = \mathbf{b}$ has solution $x_i = \dfrac{\det(A_i)}{\det(A)}$, where A_i is the matrix obtained form A by replacing the ith column of A by $\mathbf{b}$.

ALGORITHMS

1. (p 367). Determinants can be computed via reduction to triangular form. (Keep track of what each elementary operation does to the determinant.)

2. (p 374). Determinants can be computed via expansion by cofactors.

3. (p 334). To find the area of a parallelogram in $\mathbf{R}^2$:
(i) Find vectors $\mathbf{u}, \mathbf{v}$ that determine the sides of the parallelogram.
(ii) Find the absolute value of the determinant of the 2×2 matrix with columns $\mathbf{u}, \mathbf{v}$.
OR
(i) Put the coordinates of 3 vertices into the 3×3 matrix given on p 377.
(ii) Find the absolute value of the determinant of the matrix.

4. (p 348). To find the volume of a parallelepiped in $\mathbf{R}^3$:
(i) Find vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ that determine the edges of the parallelepiped.
(ii) Find the absolute value of the determinant of the 3×3 matrix with columns $\mathbf{u}, \mathbf{v}, \mathbf{w}$.
Note: By Ex 14 p 389 this is the absolute value of $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$, which computes the volume–see p 231.

USEFUL EXERCISES

(p 372 Ex #10). If k is a scalar and A is an $n \times n$ matrix, then $\det(kA) = k^n \det(A)$.

(p 372 Ex #19). *Matrices in block form:* If A and B are square matrices, then $\begin{vmatrix} A & 0 \\ C & B \end{vmatrix} = |A| \cdot |B|$.

(p 384 Ex #14). If A is an $n \times n$ matrix, then $\det(\mathrm{adj}(A)) = [\det(A)]^{n-1}$.

Chapter 7: Eigenvalues and Eigenvectors

DEFINITIONS

7.1 (pp 394, 399). Let $L : V \to V$ be a linear transformation. The *real number* λ is an **eigenvalue** of L if there is a *nonzero* vector $\mathbf{x}$ in V for which $L(\mathbf{x}) = \lambda\mathbf{x}$. In this case $\mathbf{x}$ is an **eigenvector** of L (with associated eigenvalue λ).

Note: this means that L maps the line determined by $\mathbf{x}$ back into itself.

Def (p 407, Ex 14). If λ is an eigenvalue of the linear transformation $L : V \to V$, then the **eigenspace** associated with λ is $\{\mathbf{x} \text{ in } V \mid L(\mathbf{x}) = \lambda\mathbf{x}\}$.

7.2 (p 401). The **characteristic polynomial** of a square matrix A is $\det(\lambda I - A)$.

7.3 (p 410). A linear transformation $L : V \to V$ is **diagonalizable** if it is possible to find a basis S for V such that the matrix $M_{S \leftarrow S}(L)$ of L with respect to S is a diagonal matrix.

Def (p 413). A square matrix is **diagonalizable** if it is similar to a diagonal matrix.

7.4 (p 430). A square matrix A is **orthogonal** if $A^{-1} = A^T$.

THEOREMS

7.1 (p 402). The eigenvalues of a matrix are the real roots of its characteristic polynomial.

7.2 (p 411). Similar matrices have the same eigenvalues.

7.3 (p 412). A linear transformation $L : V \to V$ is diagonalizable if and only if it is possible to find a basis for V that consists of eigenvectors of L.

7.5 (p 415). A square matrix is diagonalizable if its characteristic polynomial has distinct real roots.

7.6 (p 427). The characteristic polynomial of a symmetric matrix has real roots.

7.7 (p 428). For a symmetric matrix, eigenvectors associated with distinct eigenvalues are orthogonal.

7.8 (p 431). A square matrix is orthogonal if and only if its columns (rows) are orthonormal.

7.9 (p 431). Let A be a symmetric matrix. Then there exists an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix. The entries of $P^{-1}AP$ are the eigenvalues of A.

ALGORITHMS

1. (p 406) To find the eigenvalues and eigenvectors of the matrix A:
(i) Find the real solutions of the characteristic equation $\det(\lambda I - a) = 0$.
(ii) For each value of λ from (i), solve the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

2. (p 436) To diagonalize the matrix A:
(i) Find the eigenvalues of A by finding the real roots of its characteristic polynomial $\det(\lambda I - A)$.
(ii) For each eigenvalue λ, find a basis for its eigenspace by solving the equation $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

Note: The diagonalization process fails if the characteristic polynomial has a complex root or if it has a real root of multiplicity k whose eigenspace has dimension $< k$.

(iii) $P^{-1}AP = D$ is a diagonal matrix if the columns of P are the basis vectors found in (ii) and D is the matrix whose diagonal entries are the eigenvalues of A (in exactly the same order).

3. (p 436) To diagonalize the *symmetric* matrix A:
Follow the steps in **2**, but use the Gram-Schmidt process to find an *orthonormal* basis for each eigenspace.

Note: The diagonalization process always works, and $P^{-1} = P^T$ since P is orthogonal.