Definition: A vector space V is said to be the *direct sum* of subspaces V_1 and V_2, denoted by $V = V_1 \oplus V_2$, if each vector $v \in V$ can be written uniquely in the form $v = v_1 + v_2$, for $v_1 \in V_1$ and $v_2 \in V_2$.

1. Let V be a vector space, and let $T \in L(V, V)$. Prove that if $T^2 = T$, then $V = n(T) \oplus T(V)$.

2. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation.
   (a) Prove that if V is a linear manifold in $\mathbf{R}^n$, then $T(V)$ is a linear manifold in $\mathbf{R}^m$.
   (b) Show that if S is the directing space of V, then $\dim(T(V)) = \dim(V) - \dim(S \cap n(T))$.

3. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Prove that if V is a linear manifold of $\mathbf{R}^m$, and $T^{-1}(V) \neq \emptyset$, then $T^{-1}(V) = \{x \in \mathbf{R}^n \mid T(x) \in V\}$ is a linear manifold of $\mathbf{R}^n$.