Math 423 Homework 7 Due Wednesday, 3/28/07

20 points: This homework is an investigation of some of the properties of the "adjoint" A^* of an $n \times n$ matrix A. In upper level texts this is often called the *classical adjoint* to distinguish it from another definition given for matrices over the complex numbers.

Let $|A|$ denote the determinant of A, and let I_n denote the $n \times n$ identity matrix. Then the defining equations for the classical adjoint are $AA^* = |A| \cdot I_n$ and $A^*A = |A| \cdot I_n$.

I suspect that some of these properties are going to be hard to prove. If you aren't able to find (or prove) a general result for all $n \times n$ matrices, see what you can say in the 2×2 and 3×3 cases, and in the case of an invertible matrix. Remember that if A is invertible, then $A^* = |A| \cdot A^{-1}$.

(1) Is the classical adjoint unique in any sense (like the inverse)? That is, does $AX = |A| \cdot I_n$ imply that $X = A^*$?

(2) Is it true that $(AB)^* = B^*A^*$?

(The model for this question is the corresponding formulas for the inverse and the transpose.)

(3) Is it true that $(A + B)^* = A^* + B^*$?

(The model for this question is the corresponding formula for the transpose.)

(4) Is it true that $(\lambda A)^* = \lambda^n A^*$?

(The model for this question is the formula $|\lambda A| = \lambda^n |A|$.)

(5) Is it true that $(A^{-1})^* = (A^*)^{-1}$?

(The model for this question is the formula $(A^{-1})^t = (A^t)^{-1}$.)

(6) Is it true that $(A^t)^* = (A^*)^t$?

(7) If A is symmetric, is A^* also symmetric?

(8) If A is skew-symmetric, is A^* also skew-symmetric? If not, when is this true, and what *can* you say about the classical adjoint of a general skew-symmetric matrix?

As we showed in class, it is true in the 2×2 case, but not the 3×3 case. To develop a conjecture about the general case, compute the classical adjoint of a skew-symmetric 3×3 matrix.

From a mathematician's point of view, the answer to each question should be a proof or a counterexample, or perhaps a correction, together with a proof. It the result holds in certain special cases, this again calls for a proof and perhaps some examples showing that your statement is as good as possible.