MATH 423 TEST I 2/13/98

Professor John Beachy

Each question is worth 20 points. The field of real numbers will be denoted by $\mathbf{R}$.
For use in questions 4 and 5, here is the definition of a direct sum: A vector space V is the direct sum of subspaces W_1 and W_2, denoted by $V = W_1 \oplus W_2$, if $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$.

1. Give careful statements of the following definitions: linear dependence; generating set (or spanning set); basis; linear transformation.

2. (a) Let V be the vector space P_2 of all polynomials over $\mathbf{R}$ of degree ≤ 2. Show that the set $\{x^2 + x + 1,\ x + 1,\ 1\}$ is a basis for V.

   (b) Let (a, b, c) be any nonzero vector in $\mathbf{R}^3$. Let $P = \{(x, y, z) \in \mathbf{R}^3 \mid ax+by+cz = 0\}$. Find a basis for P.

3. Prove Theorem 1.8 from the text: Let S be a linearly independent subset of a vector space V, and let v be an element of V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in span(S)$.

4. Let V be the vector space of all functions defined on $\mathbf{R}$ that take values in $\mathbf{R}$. Let W_1 be the set of all even functions in V (that is, all functions $f(x)$ with $f(-x) = f(x)$, for all $x \in \mathbf{R}$). Let W_2 be the set of all odd functions in V (that is, all functions $f(x)$ with $f(-x) = -f(x)$, for all $x \in \mathbf{R}$).

   (a) Show that W_1 is a subspace of V.

   (b) Given that W_2 is a subspace of V, show that $V = W_1 \oplus W_2$.

5. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation.

   Note: Parts (a) and (b) are proofs from the text.

   (a) Show that $N(T) = \{x \in V \mid T(x) = 0\}$ is a subspace of V.

   (b) Show that $R(T) = \{y \in W \mid y = T(x) \text{ for some } x \in V\}$ is a subspace of W.

   (c) Suppose that $W = V$, and $T^2 = T$. Show that $V = N(T) \oplus R(T)$.