Choose 5 of the following 6 questions. Each one is worth 20 points. If you cannot complete a proof, you can get partial credit by stating the relevant definitions and results. In the computational examples, be sure to justify your statements. Be brief and to the point.

1. List the possible abelian groups of order 200 (up to isomorphism). To which of these is the group $\mathbf{Z}_{500}^{\times}$ isomorphic?

2. Prove that there are no simple groups of order 96 or 135.

3. Prove the First Isomorphism Theorem. That is, let G be a group, let N be a normal subgroup of G, and let H be any subgroup of G. Prove that $H \cap N$ is a normal subgroup of H, and $(HN)/N \cong H/(H \cap N)$. *Note:* Assume that HN is a subgroup of G.

4. Let G be a group of order mp^{α}, where p is prime and $p \not| m$. Assume that the Sylow p-subgroups of G are known to be conjugate. Write out the proof that the number of Sylow p-subgroups of G is a divisor of m and $\equiv 1 \mod p$. (This is part of the proof of the Sylow theorems.)

5. Let D_{15} be the dihedral group given by generators a and b with relations $a^{15} = 1$, $b^2 = 1$, and $ba = a^{-1}b$. Find the conjugacy classes of D_{15}, and find all of its Sylow subgroups.

6. Let G be the Frobenius group F_{20} of matrices of the form $\begin{bmatrix} 1 & 0 \\ x & a \end{bmatrix}$ such that $x \in \mathbf{Z}_5$ and $a \in \mathbf{Z}_5^{\times}$. Find a composition series for G. Find the descending series of commutator subgroups of G.

Group Theory – choose 2 of the first 3 (25 points each)

1. (a) State the Sylow theorems.

   (b) Prove that there is no simple group of order 105.

2. (a) State the fundamental theorem of finite abelian groups.

   (b) Suppose that G is an abelian group which has 8 elements of order 3, 18 elements of order 9, and no other elements besides the identity. Find (with proof) the decomposition of G as a direct product of cyclic groups.

3. (a) Write out the definition of a solvable group.

   (b) Prove the theorem which states that any finite p-group is solvable, where p is a prime number.

Fields and Galois Theory – (30 points each)

4. (a) Let F be a splitting field for $f(x) \in K[x]$. Define $\mathrm{Gal}(F/K)$.

   (b) Find the Galois group of the polynomial $x^5 - 1$ over $\mathbf{Q}$.

   Hint: Use the theorems!

5. (a) State the fundamental theorem of Galois theory.

   (b) Let F be a splitting field for $f(x) \in K[x]$, and let E be an intermediate field with $K \subseteq E \subseteq F$. If $\mathrm{Gal}(F/K) = S_3$ and E is a splitting field over K with $E \neq K$, what can you say about $[E : K]$?

6. Prove the theorem which states that if F is the splitting field of $x^n - 1$ over a field K of characteristic zero, then $\mathrm{Gal}(F/K)$ is an abelian group.

7. Let K be a finite field, and let F be an extension of K with $[F : K] = m$. Prove the theorem which states that $\mathrm{Gal}(F/K)$ is a cyclic group of order m, and give an explicit formula for the generator of the group.

8. Let $F = \mathbf{Q}(\sqrt[4]{2}, i)$ and let $K = \mathbf{Q}(i)$.

   (a) Find $\mathrm{Gal}(F/K)$.

   (b) Find all subfields between F and K. Which ones are normal extensions of K?

   Rules: You need to do this from scratch, not using facts from the example in the text or from your homework problem. Use the theorems–the answer doesn't require much computation.