MATH 520 Homework III 9/12/2007
Prof. John Beachy Due 9/21/2007

1. (2.1, p67 #8): Let X be an abelian group, and let G be any group. Find necessary and sufficient conditions to guarantee that the semidirect product $X \rtimes G$ is abelian.

2. (2.1, p67 #14): Show that S_4 is the semidirect product of a normal subgroup of order 4 by a subgroup of order 6.

3. (2.1, p67 #15): Find the center of the Frobenius group F_{20} defined in Exercise 20 of Section 1.4.

4. (2.1, p67 #16): Show that the Frobenius group F_{20} can be defined by generators and relations as follows.

Let $a = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(a) Show that $o(a) = 5$, $o(b) = 4$, and $ba = a^2 b$.

(b) Show that each element of F_{20} can be expressed in the form $a^i b^j$ for $0 \le i \le 4$ and $0 \le j \le 3$.

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1. (2.2, p75 #8): Find the conjugacy classes of the quaternion group Q_8 defined in Example 1.1.5, and write out its conjugacy class equation.

2. (2.2, p75 #9): Let the dihedral group D_n be given by elements a of order n and b of order 2, where $ba = a^{-1}b$. Show that a^m is conjugate to only itself and a^{-m}, and that $a^m b$ is conjugate to $a^{m+2k}b$, for any integer k.

3. (2.2, p75 #10): Show that the Frobenius group F_{20} (defined in Exercise 20 of Section 1.4) is isomorphic to the subgroup of S_5 generated by the permutations $(1, 2, 3, 4, 5)$ and $(2, 3, 5, 4)$. Use this fact to help in finding the conjugacy classes of F_{20}, and its conjugacy class equation.

4. (2.2, p75 #14): Determine the conjugacy classes of the alternating group A_5, and use this information to show that A_5 is a simple group.

5. (2.3, p82, #2): Let H and K be subgroups of the group G, and let S be the set of left cosets of K. Define a group action of H on S by setting $a \cdot (xK) = axK$, for all $a \in H$ and $x \in G$. By considering the orbit of K under this action, show that $|HK| = \dfrac{|H|\,|K|}{|H \cap K|}$. (This gives a more sophisticated proof of Exercise 14.27.)

6. (2.3, p82, #4): Let G be any non-abelian group of order 6. By Cauchy's Theorem, G has an element, say a, of order 2. Let $H = \langle a \rangle$, and let S be the set of left cosets of H.

(a) Show that H is not normal in G.

Hint: If H is normal, then $H \subseteq Z(G)$, and it can then be shown that G is abelian.

(b) Use Exercise 1 and part (a) to show that G must be isomorphic to $\mathrm{Sym}(S)$. Thus any non-abelian group of order 6 is isomorphic to S_3.

7. (2.3, p83, #8): If G is a finite group of order n and p is the least prime such that $p|n$, show that any subgroup of index p is normal in G.

8. (2.3, p83, #14): Show that the group of rigid motions of a cube is isomorphic to S_4.
Hint: Define an action of the group of rigid motions of a cube on the set of four pairs of opposite vertices, and consider the corresponding group homomorphism.