MATH 520 | Homework IX | 11/19/07
Prof. John Beachy | Due 12/07/07

4.3 #1: In Example 4.3.3 use a direct calculation to verify that the subfield fixed by $\langle \alpha^3 \beta \rangle$ is $\mathbf{Q}(\sqrt[4]{2} - i\sqrt[4]{2})$.

4.3 #2: In Example 4.3.3 determine which subfields are conjugate, and in each case find an automorphism under which the subfields are conjugate.

4.4 #3: Find the Galois group of $x^5 - 1$ over $\mathbf{Q}$.

4.4 #4: Find the Galois group of $x^9 - 1$ over $\mathbf{Q}$.

4.4 #7: Let H be a subgroup of S_p, where p is prime. Show that if H contains a transposition and a cycle of length p, then $H = S_p$.

4.4 #8: Prove that if $f(x) \in \mathbf{Q}[x]$ is irreducible of prime degree p and has exactly two non-real roots in $\mathbf{C}$, then the Galois group of $f(x)$ over $\mathbf{Q}$ is S_p.

The remaining problems are worth 10 points each, instead of the usual 5 points.

IX #1. Let the field F be a finite, normal, separable extension of the field K. Suppose that the Galois group of F over K is cyclic of order 50. Find how many distinct fields E there are with $K \subseteq E \subseteq F$, and how many of these are normal extensions of K.

IX #2. Find the Galois group of $x^3 - 7$ over $\mathbf{Q}$.

IX #3. Let $u = \sqrt{2 + \sqrt{2}}$. Let $f(x)$ be the minimal polynomial of α over $\mathbf{Q}$, and let F be the splitting field for $f(x)$ over $\mathbf{Q}$. Prove that $\mathrm{Gal}(F/\mathbf{Q})$ is cyclic of order 4. Find all fields E with $\mathbf{Q} \subseteq E \subseteq F$.

IX #4. Show that the Galois group of $x^5 - 2$ over $\mathbf{Q}$ is F_{20}. (This exercise is in Dummit and Foote.)

IX #5. This exercise (from Dummit and Foote) shows that $\mathrm{Gal}(\mathbf{R}/\mathbf{Q}) = \{1_{\mathbf{R}}\}$.

(a) Let $\alpha \in \mathrm{Gal}(\mathbf{R}, \mathbf{Q})$. Show that α maps squares to squares, and maps positive reals to positive reals. Conclude that $a < b$ implies $\alpha(a) < \alpha(b)$ for all $a, b \in \mathbf{R}$.

(b) Prove that $-\frac{1}{m} < a - b < \frac{1}{m}$ implies that $-\frac{1}{m} < \alpha(a) - \alpha(b) < \frac{1}{m}$ for all $a, b \in \mathbf{R}$. Conclude that α is a continuous function on $\mathbf{R}$.

(c) Prove that any continuous function from $\mathbf{R}$ to $\mathbf{R}$ that is the identity on $\mathbf{Q}$ is the identity mapping.