1. State 3 of the following theorems.
   (1) The Hilbert basis theorem
   (2) The Artin-Wedderburn theorem
   (3) Maschke’s theorem
   (4) The Krull-Schmidt theorem
   (5) The Chinese remainder theorem
   (6) The fundamental structure theorem for finitely generated modules over a principal ideal domain

2. Solve Part A or Part B.
   Part A. Let $R$ be a ring. For elements of $R$,
   (a) state the following definitions: nilpotent element, idempotent element, regular element, zero divisor.
   (b) Which of the following sets of elements of $R$ are closed under addition? (Give a proof or a counterexample.)
      The set of all nilpotent elements; the set of all idempotent elements; the set of all regular elements; the set of all zero divisors.
   Part B. Let $D$ be an integral domain with prime ideal $P$.
   (a) Describe the construction of the localization $D_P$ of $D$ at $P$.
   (b) Describe the connection between the ideals of $D$ and those of $D_P$.
   (c) Prove that if $D$ is a Noetherian ring, then so is the localization $D_P$.

3. Solve Part A or Part B.
   Part A. Let $R$ be a ring, and let $M$ be a left $R$-module.
   (a) Complete this definition: $M$ is a simple module if . . .
   (b) Prove that $M$ is simple if and only if $M = Rm$, for all nonzero $m \in M$.
   (c) Prove that $M$ is simple if and only if $M \cong R/A$ for a maximal left ideal $A \subseteq R$.
   (d) State and prove Schur’s lemma.
   Part B. Let $R$ be a ring, and let $M$ be a left $R$-module.
   (a) Complete this definition: $M$ is an Artinian module if . . .
   (b) Prove that if $N$ is a submodule of $M$, then $M$ is Artinian if and only if both $N$ and $M/N$ are Artinian.
   (c) Prove that a left Artinian ring with no nonzero divisors of zero is a division ring.

4. Solve Part A or Part B.
   Part A
   (a) State the definition of the tensor product of two modules.
   (b) Show that it follows from the definition that the tensor product is unique up to isomorphism.
(c) Show that $\mathbb{Z}_n \otimes \mathbb{Z} \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_n$. (Hint: you might show that $1 \otimes 1$ is a generator.)

Part B

(a) State the definitions of projective module and injective module.
(b) State Baer’s criterion for injectivity.
(c) Prove that $\mathbb{Z}_n$ is injective as a left module over $\mathbb{Z}_n$. (That is, that $\mathbb{Z}_n$ is a self-injective ring.)

Extra credit (20 points):

Find the prime and maximal ideals of the ring of $2 \times 2$ lower triangular matrices over $\mathbb{Z}$.
(Justify your answer.)

OR

Write out the solution to one of your favorite problems. (Points will be adjusted for difficulty.)